Complex Anlysis Quiz-1 Solution

Oct. 19, 2018 DCChang

1. (10pts)

$$
\begin{aligned}
& e^z = (1+i)/\sqrt{2} \\
\iff & e^x \cos y + i e^x \sin y = (1+i)/\sqrt{2} \\
\iff & x = 0 \text{ and } y = \frac{\pi}{4} + 2k\pi,\ k = 0, \pm 1, \pm 2, \ldots \\
\iff & z = \left(\frac{\pi}{4} + 2k\pi\right) i,\ k = 0, \pm 1, \pm 2, \ldots
\end{aligned}
$$

2.

(a) (5 pts)

$$z = \log 2i = \operatorname{Log} 2 + i\left(\frac{\pi}{2} + 2k\pi\right), \quad k = 0, \pm 1, \pm 2, \ldots$$

(b) (10 pts)

$$e^z = \frac{-1 + \sqrt{-3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$
\begin{aligned}
z &= \log\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) \\
&= \operatorname{Log}\left|-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right| + i \arg\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) \\
&= i\left(\pm\frac{2\pi}{3} + 2k\pi\right), \quad k = 0, \pm 1, \ldots
\end{aligned}
$$

3.

(a) (10 pts)

From the equation

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i},$$

we deduce that

$$e^{2iw} - 2ize^{iw} - 1 = 0. \tag{6}$$

Using the quadratic formula we can solve Eq. (6) for e^{iw}:

$$e^{iw} = iz + (1 - z^2)^{1/2},$$

where, of course, the square root is two-valued. Formula (5) now follows by taking logarithms. ■

(b) (10 pts)

$$\begin{aligned} z = \sin^{-1} 2 &= -i \log\left[2i + (-3)^{\frac{1}{2}}\right] \\ &= -i\mathrm{Log}\,(2 \pm \sqrt{3}) + \arg[(2 \pm \sqrt{3})i] \\ &= -i\mathrm{Log}\,(2 \pm \sqrt{3}) + \frac{\pi}{2} + 2k\pi, \quad k = 0, \pm 1, \ldots \end{aligned}$$

Now observe that since $(2+\sqrt{3})(2-\sqrt{3}) = 1$,
$0 = \mathrm{Log}\,[(2+\sqrt{3})(2-\sqrt{3})] = \mathrm{Log}\,(2+\sqrt{3}) + \mathrm{Log}\,(2-\sqrt{3})$ so that

$$\mathrm{Log}(2-\sqrt{3}) = -\mathrm{Log}(2+\sqrt{3}).$$

Therefore,

$$z = \pm i\mathrm{Log}(2+\sqrt{3}) + \frac{\pi}{2} + 2k\pi \quad k = 0, \pm 1, \ldots$$

4. (15 pts)

$$\lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z}$$

$$\begin{aligned} &= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\overline{z}_0 + \overline{\Delta z}) - z_0 \overline{z}_0}{\Delta z} \\ &= \lim_{\Delta z \to 0} \left(\overline{z}_0 + \frac{\overline{\Delta z}}{\Delta z} z_0 + \overline{\Delta z}\right) = \begin{cases} \overline{z_0} + z_0 & \text{if } \Delta z = \Delta x \\ \overline{z_0} - z_0 & \text{if } \Delta z = i\Delta y \end{cases} \end{aligned}$$

If $z_0 = 0$, then the difference quotient is

$$\lim_{\Delta z \to 0} (0 + 0 + \overline{\Delta z}) = 0.$$

5. (20 pts)
(Method 1)

$z = r(\cos\theta + i\sin\theta)$

$\Delta z = (r+\Delta r)\,[\cos(\theta+\Delta\theta) + i\sin(\theta+\Delta\theta)]$
$\quad - r(\cos\theta + i\sin\theta) = (r+\Delta r)e^{i(\theta+\Delta\theta)} - re^{i\theta}$

differentiate $f(z)$ along r-axis while setting $\Delta\theta = 0$

$f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

$= \lim\limits_{\Delta r \to 0} \left(\dfrac{u(r+\Delta r, \theta) - u(r,\theta)}{\Delta r e^{i\theta}} + i\, \dfrac{v(r+\Delta r, \theta) - v(r,\theta)}{\Delta r \cdot e^{i\theta}} \right)$

$= e^{-i\theta}(u_r + i v_r)$. #

② along θ-axis while setting $\Delta r = 0$

$$\Delta z = re^{i(\theta+\Delta\theta)} - re^{i\theta} = re^{i\theta}(e^{i\Delta\theta} - 1)$$

$$= re^{i\theta} \cdot i\sin\Delta\theta \approx re^{i\theta} \cdot i\Delta\theta$$

$$f'(z) = \lim_{z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta\theta \to 0} \left(\frac{u(r,\theta+\Delta\theta) - u(r,\theta)}{re^{i\theta} \cdot i\Delta\theta} + i \frac{v(r,\theta+\Delta\theta) - v(r,\theta)}{re^{i\theta} \cdot i\Delta\theta} \right)$$

$$= \frac{1}{re^{i\theta}}(v_\theta - i u_\theta) \#$$

From 1 and 2,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

(Method 2)

$z = re^{i\theta} \Longrightarrow x = r\cos\theta$ and $y = r\sin\theta$ and

$$f(z) = u(x(r,\theta), y(r,\theta)) + iv(x(r,\theta), y(r,\theta))$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

Similar applications of the chain rule yield

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y} r\cos\theta$$
$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta$$
$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y} r\cos\theta$$

Replace the partial derivatives on the right sides of the equations for $\frac{\partial u}{\partial r}$ and $\frac{\partial v}{\partial r}$ by their Cauchy-Riemann counterparts to obtain:

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta = \frac{1}{r}\frac{\partial v}{\partial \theta}$$
$$\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

6. (10 pts)

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} = \frac{\partial v}{\partial y} \Rightarrow$$
$$v(x,y) = \int \frac{x}{x^2+y^2}dy = \tan^{-1}\left(\frac{y}{x}\right) + \psi(x)$$
$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} = -\frac{\partial v}{\partial x} = \frac{y}{x^2+y^2} - \psi'(x) \Rightarrow \psi(x) = a$$

Thus, $v(x,y) = \tan^{-1}\left(\frac{y}{x}\right) + a.$

7.

(a) (5 pts)

$$\left(\frac{2i}{1+i}\right)^{1/6} = (1+i)^{1/6} = \sqrt[12]{2}\exp\left(i\frac{\pi/4+2k\pi}{6}\right), k = 0,1,2,3,4,5$$

(b) (5 pts)

$$z = 1 \pm \sqrt{1-i} = 1 \pm 2^{1/4}\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$$