Complex Anlysis Quiz-1 Solution
18 Oct., 2019 DCChang

1. (10 %)

$$z = x + iy$$

$$|z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 - z_2)} = z_1\bar{z}_1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_2\bar{z}_2$$
$$= |z_1|^2 - (z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) + |z_2|^2$$
$$= |z_1|^2 - 2\,\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2 \quad \#$$

2. (10% +10 %)

(a)

**Solution.** We can express the integrand as

$$\cos^4\theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^4 = \frac{1}{2^4}\left(e^{i\theta} + e^{-i\theta}\right)^4,$$

and expanding via the binomial formula gives

$$\begin{aligned}\cos^4\theta &= \frac{1}{2^4}\left(e^{4i\theta} + 4e^{3i\theta}e^{-i\theta} + 6e^{2i\theta}e^{-2i\theta} + 4e^{i\theta}e^{-3i\theta} + e^{-4i\theta}\right)\\ &= \frac{1}{2^4}\left(e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}\right)\\ &= \frac{1}{2^4}(6 + 8\cos 2\theta + 2\cos 4\theta).\end{aligned}$$

(b)

$$\begin{aligned}\int_0^{2\pi}\cos^4\theta\,d\theta &= \int_0^{2\pi}\frac{1}{2^4}(6 + 8\cos 2\theta + 2\cos 4\theta)\,d\theta\\ &= \frac{1}{2^4}[6\theta + 4\sin 2\theta + \frac{1}{2}\sin 4\theta]\Big|_0^{2\pi} = \frac{6}{2^4}2\pi = \frac{3}{4}\pi.\end{aligned}$$

3. (5% +5 %)

(a)

$$(-16)^{1/4} = 2\exp\left(i\frac{\pi + 2k\pi}{4}\right),\ k = 0, 1, 2, 3$$

(b)

$$\left(\frac{2i}{1+i}\right)^{1/6} = (1+i)^{1/6} = \sqrt[12]{2}\exp\left(i\frac{\pi/4 + 2k\pi}{6}\right),\ k = 0, 1, 2, 3, 4, 5$$

4. (10%)

$z = 2 - i,\ 1 - i$

5. (15%)

path(i): along the y-axis direction, $x=0$

$$\lim_{z\to 0} \frac{x^2+iy^2}{|z^2|} = \lim_{y\to 0} \frac{iy^2}{|y^2|} = i$$

path(ii): along the x-axis direction, $y=0$

$$\lim_{z\to 0} \frac{x^2+iy^2}{|z^2|} = \lim_{x\to 0} \frac{x^2}{|x^2|} = 1$$

$\Rightarrow f(z)$ is not continuous at $z_0=0$ ※

6. (15%)

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy - 1.$$

$$v(x, y) = 3x^2y - y^3 + \psi(x).$$

$$\frac{\partial v}{\partial x} = 6xy + \psi'(x) = 6xy - 1.$$

$$v(x, y) = 3x^2y - y^3 - x + a,$$

$$f(z) = x^3 - 3xy^2 + y + i\left(3x^2y - y^3 - x + a\right)$$

7. (20%)

$$\frac{\partial u}{\partial x} = 2e^{x^2-y^2}[x\cos(2xy) - y\sin(2xy)] = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -2e^{x^2-y^2}[y\cos(2xy) + x\sin(2xy)] = -\frac{\partial v}{\partial x}$$

**$f$ is entire because these first partials exist and are continuous for all $x$ and $y$.**

$$\begin{aligned} f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} &= 2e^{x^2-y^2}(x+iy)[\cos(2xy) + i\sin(2xy)] \\ &= 2e^{(x^2-y^2)}e^{i2xy}(x+iy) \\ &= 2ze^{z^2} \end{aligned}$$