Complex Analysis 2016 Final D.C. Chang 2017/1/16

#1. (a) $f(z) = z^3 + 3z + 1$ is analytic in the whole plane

$f'(z) = 3z^2 + 3 = 0 \Rightarrow z = \pm i$

$f(z)$ is conformal mapping except $z = \pm i$. #

(b) $f(z) = z - e^{-z} + i$ is analytic in the whole plane

$f'(z) = 1 + e^{-z} = 0 \Rightarrow e^z + 1 = 0$

$e^z = -1 = e^{\ln(-1)} = e^{i(\pi + 2n\pi)}$, $n = 0, \pm 1, \pm 2, \dots$

$\therefore f(z)$ is conformal except $z = i(2n+1)\pi$, $n = 0, \pm 1, \pm 2, \dots$

(or $z = -i(2n+1)\pi$, $n = 0, \pm 1, \pm 2, \dots$) #

(c) $f(z) = \tan z = \dfrac{\sin z}{\cos z}$

$f(z)$ is not analytic when $\cos z = 0$, i.e.

$z = \frac{\pi}{2} + n\pi = \frac{2n+1}{2}\pi$, $n = 0, \pm 1, \pm 2, \dots$

$f'(z) = \left(\dfrac{\sin z}{\cos z}\right)' = \dfrac{\sin^2 z + \cos^2 z}{\cos^2 z} = \dfrac{1}{\cos^2 z} \neq 0$

$\therefore f(z)$ is conformal mapping except $z = \frac{2n+1}{2}\pi$, $n = 0, \pm 1, \pm 2, \dots$ #

#2: By implicit formula,

$$\frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1} = \frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1}$$

$(z: -i, 1, \infty) \xrightarrow{\text{mapping}} (w; 1, i, -1)$

$z_3 \to \infty \quad \therefore \dfrac{z - z_1}{z_2 - z_1} = \dfrac{w - w_1}{w - w_3} \cdot \dfrac{w_2 - w_3}{w_2 - w_1}$

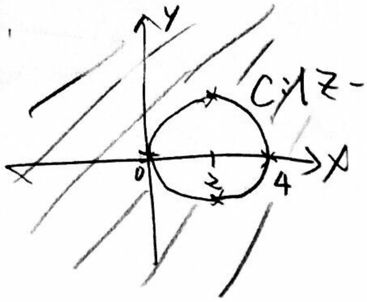
$\Rightarrow \dfrac{z + i}{1 + i} = \dfrac{(w-1)(i+1)}{(w+1)(i-1)} = -i \cdot \dfrac{w-1}{w+1}$

$\dfrac{w-1}{w+1} = \dfrac{iz - 1}{1 + i} \Rightarrow (iz - 1)w + iz - 1 = (1+i)w - (1+i)$

$\therefore (iz - 2 - i)w = 1 - iz - 1 - i = -i(z+1)$

$\Rightarrow w = \dfrac{-i(z+1)}{iz - 2 - i} = -\dfrac{z+1}{z + 2i - 1}$ #

3.



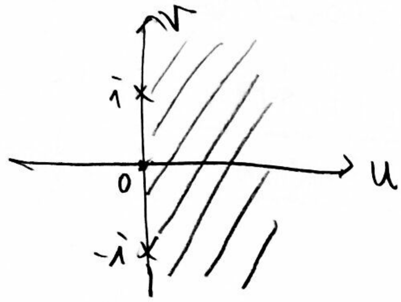
$W = \frac{z}{z-4}$

$\because$ The pole of ω is 4 which is on C,

$\therefore$ ω is a stright line

pick $z: (0, 2-2i, 4, 2+2i)$

$\Rightarrow W: (0, i, \infty, -i)$ $\swarrow W = \frac{z}{z-4}$



mapping onto $\underline{Re\{w\} = u > 0}$.

i.e. $\underline{\text{right half-plane}}$. ※

4. Let $g(z) = (\sin z - z + z^2)^2$, $g(0) = 0$.

$g'(z) = 2(\cos z - 1 + 2z)(\sin z - z + z^2)$, $g'(0) = 0$

$g''(z) = 2(-\sin z + 2)(\sin z - z + z^2) + 2(\cos z - 1 + 2z)^2$

$g''(0) = 0$.

$g'''(z) = 2(-\cos z)(\sin z - z + z^2) + 2(-\sin z + 2)(\cos z - 1 + 2z)$
$\quad + 4(-\sin z + 2)(\cos z - 1 + 2z)$

$g'''(0) = 0$

$g^{(4)}(z) = 2\sin z(\sin z - z + z^2) + 2(-\cos z)(\cos z - 1 + 2z)$
$\quad + 2(-\cos z)(\cos z - 1 + 2z) + 2(-\sin z + 2)^2$
$\quad + 4(-\cos z)(\cos z - 1 + 2z) + 4(-\sin z + 2)^2$

$g^{(4)}(0) \neq 0$ $\therefore$ $\underline{\text{the order of } f(z) \text{ is } 4}$. ※

or: $\because g(0) = 0$, for $z \to 0$, $\sin z \approx z$

$\therefore f(z) \approx \frac{1}{(z - z + z^2)^2} = \frac{1}{z^4}$ $\Rightarrow$ $\underline{\text{the order of } f(z) \text{ is } 4}$ ※

5. $f(z) = \dfrac{z^2(z+i)^4(z-3)^6 e^{z^2}}{3(z-1)^3(2z-7)^5}$ inside $D_2^+(0)$ ③

$Z_f = 2+4 = 6$

$P_f = 3$

By Argument principle $\oint_{D_2^+(0)} \dfrac{f'(z)}{f(z)} dz = 2\pi i \underline{(Z_f - P_f) = 6\pi i}$ #

6. (a) $\because \tan z = \dfrac{\sin z}{\cos z} = \dfrac{z(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots)}{1-\frac{z^2}{2}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots} = z(1+az^2+bz^4+\cdots)$

$\therefore \tan\left(\frac{1}{2z}\right) = \frac{1}{2z}\left[1+a\cdot\left(\frac{1}{2z}\right)^2+b\cdot\left(\frac{1}{2z}\right)^4+\cdots\right]$

$e^{\frac{1}{z}} = 1+\frac{1}{z}+\frac{1}{2}\left(\frac{1}{z}\right)^2+\frac{1}{3!}\left(\frac{1}{z}\right)^3+\cdots$

$\operatorname{Res}_{z=0}\left[e^{\frac{1}{z}}\cdot\tan\left(\frac{1}{2z}\right)\right] = \frac{1}{2}$

$\oint_{|z|=1} e^{\frac{1}{z}}\cdot\tan\left(\frac{1}{2z}\right)dz = 2\pi i \operatorname{Res}_{z=0}\left[e^{\frac{1}{z}}\tan\left(\frac{1}{2z}\right)\right]$

$= \pi i$ #

(b) $\oint_{|z|=2} \dfrac{e^z}{z^3+5z^2} dz$

$= \oint_{|z|=2} \dfrac{1}{z^2}\cdot\dfrac{e^z}{z+5} dz$

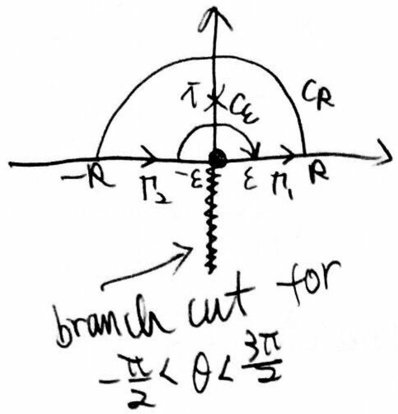
$= 2\pi i \cdot \operatorname{Res}_{z=0}\left(\dfrac{1}{z^2}\cdot\dfrac{e^z}{z+5}\right)$

$= 2\pi i\cdot\left(\dfrac{e^z}{z+5}\right)'\Big|_{z=0}$

$= 2\pi i \left.\dfrac{e^z(z+5)-e^z}{(z+5)^2}\right|_{z=0}$

$= 2\pi i\cdot\dfrac{5-1}{25} = \dfrac{8\pi i}{25}$ #

#17. Let $f(z) = \dfrac{\log z}{(z^2+1)^2}$, $z = re^{i\theta}$, $r>0$, $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ ④



$C = C_R + C_\varepsilon + \Gamma_1 + \Gamma_2$, $\oint_C f(z)dz = 2\pi i \operatorname{Res}_{z=i} f(z)$

as $\varepsilon \to 0$, $R \to \infty$

$\int_{C_R} f(z)dz = 0$. $\int_{C_\varepsilon} f(z)dz = 0$

(proof can be omitted!)

① Γ_1 : $z = re^{i\cdot 0} = r$, $dz = dr$

$$\int_{\Gamma_1} f(z)dz = \int_\varepsilon^R \frac{\log r}{(r^2+1)^2} dr$$

② Γ_2 : $z = re^{i\pi}$, $\log z = \log r + i\pi$, $dz = -dr$

$z: -R \longrightarrow -\varepsilon$

$r: R \longrightarrow \varepsilon$

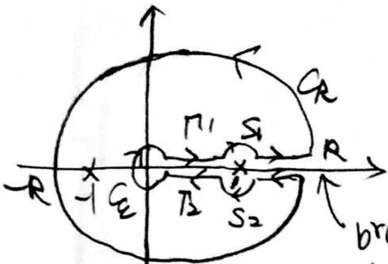
$$\int_{\Gamma_2} f(z)dz = \int_R^\varepsilon \frac{\log r + i\pi}{(r^2 e^{i2\pi}+1)^2}(-dr) = \int_0^\infty \frac{\log r}{(r^2+1)^2} dr + i\int_0^\infty \frac{\pi}{(r^2+1)^2} dr$$

$$\begin{aligned}
\oint_C f(z)dz &= 2\pi i \cdot \operatorname{Res}_{z=i} f(z) \\
&= 2\pi i \cdot \frac{d}{dz}\left(\frac{\log z}{(z+i)^2}\right)\Bigg|_{z=i} \\
&= 2\pi i \cdot \frac{\frac{1}{z}(z+i)^2 - 2\log z\cdot(z+i)}{(z+i)^4}\Bigg|_{z=i} \\
&= 2\pi i \cdot \frac{\frac{1}{z}(z+i) - 2\log z}{(z+i)^3}\Bigg|_{z=i} \\
&= 2\pi i \cdot \frac{2 - 2\log i}{-8i} = \frac{\pi(1 - i\frac{\pi}{2})}{-2} \quad (\because \log i = i\cdot\frac{\pi}{2}) \\
&= -\frac{\pi}{2} + i\cdot\frac{\pi^2}{4}
\end{aligned}$$

$$\Rightarrow 2\int_0^\infty \frac{\log r}{(r^2+1)^2} dr = -\frac{\pi}{2} \quad \therefore \underline{\int_0^\infty \frac{\log x}{(x^2+1)^2} dx = -\frac{\pi}{4}} \quad \#$$

#8 Let $f(z) = \dfrac{z^\alpha}{z^2-1}$, $z = re^{i\theta}$, $r>0$, $0<\theta<2\pi$ ⑤

$-1<\alpha<1$



$C = C_R + C_\varepsilon + \Gamma_1 + \Gamma_2 + S_1 + S_2$

$$\oint_C f(z)\,dz = 2\pi i \cdot \operatorname*{Res}_{z=-1} f(z)$$
$$= 2\pi i \cdot \left(\frac{z^\alpha}{z-1}\right)_{z=-1}$$
$$= 2\pi i \cdot \frac{(-1)^\alpha}{-2}$$
$$= -\pi i \cdot e^{i\alpha\pi}$$

$(-1)^\alpha = e^{\alpha \ln(-1)} = e^{i\alpha\pi}$

① C_R: $r = R \to \infty$, $z = Re^{i\theta}$, $dz = Ri \cdot e^{i\theta}\,d\theta$

$$\int_{C_R} f(z)\,dz \le \int_0^{2\pi} \frac{|R^\alpha \cdot e^{i\alpha\theta}|}{|R^2 e^{i2\theta} - 1|} |Rie^{i\theta}|\,d\theta$$
$$\le \lim_{R\to\infty} \frac{2\pi R^{\alpha+1}}{R^2-1} = \lim_{R\to\infty} 2\pi \cdot R^{\alpha-1} = 0, \quad \because \alpha-1<0.$$

② C_ε: $r = \varepsilon \to 0$, $z = \varepsilon e^{i\theta}$, $dz = \varepsilon i e^{i\theta}\,d\theta$

$$\int_{C_\varepsilon} f(z)\,dz \le \lim_{\varepsilon\to 0} \frac{2\pi \varepsilon^{\alpha+1}}{1-\varepsilon^2} = \lim_{\varepsilon\to 0} 2\pi \varepsilon^{\alpha+1} = 0, \quad \because \alpha+1>0.$$

③ S_1: $z = re^{i0}$, $z^\alpha = r^\alpha$

$$\int_{S_1} f(z)\,dz = -\pi i \cdot \operatorname*{Res}_{z=1} f(z) = -\pi i \left(\frac{z^\alpha}{z+1}\right)_{z=1} = -\frac{\pi i}{2}$$

④ S_2: $z = re^{i2\pi}$, $z^\alpha = r^\alpha \cdot e^{i2\alpha\pi}$

$$\int_{S_2} f(z)\,dz = e^{i2\alpha\pi} \cdot \int_{S_1} f(z)\,dz = -\frac{\pi i}{2} \cdot e^{i2\alpha\pi}$$

⑤ Γ_1: $z = re^{i0} = r$

$$\int_{\Gamma_1} f(z)\,dz = P.V. \int_0^\infty \frac{x^\alpha}{x^2-1}\,dx = I.$$

⑥ Γ_2: $z = re^{i2\pi}$, $z^\alpha = r^\alpha e^{i2\alpha\pi}$

$$\int_{\Gamma_2} f(z)\,dz = P.V. \int_\infty^0 \frac{r^\alpha \cdot e^{i2\alpha\pi}}{r^2-1}\,dr = -e^{i2\alpha\pi} \cdot I$$

$$\Rightarrow (1-e^{i2\alpha\pi}) I = \frac{\pi i}{2}(1+e^{i2\alpha\pi}) - \pi i \cdot e^{i\alpha\pi}$$
$$= \frac{\pi i}{2}(1 + e^{i2\alpha\pi} - 2e^{i\alpha\pi})$$

⑥

$$\Rightarrow \left(e^{-i\alpha\pi} - e^{i\alpha\pi}\right) I = \frac{\pi i}{2}\cdot\left(e^{-i\alpha\pi} + e^{i\alpha\pi} - 2\right)$$

$$I = \frac{\pi i}{2}\cdot\frac{2\cos\alpha\pi - 2}{-2i\sin\alpha\pi} = \frac{\pi(1-\cos\alpha\pi)}{2\sin\alpha\pi}$$ ※

#9. (a) Let $z = e^{i\theta}$, $0<\theta<2\pi$

$\cos\theta = \frac{1}{2}(z+z^{-1})$, $dz = i\cdot e^{i\theta}\cdot d\theta = iz\,d\theta$

$\cos 2\theta = \frac{1}{2}(z^2+z^{-2})$

$$I = \int_0^{2\pi}\frac{\cos 2\theta}{13-12\cos\theta}\,d\theta = \oint_{|z|=1}\frac{\frac{1}{2}(z^2+z^{-2})}{13-6(z+z^{-1})}\frac{1}{iz}\,dz$$

Let $f(z) = \frac{1}{iz}\cdot\frac{\frac{1}{2}(z^2+z^{-2})}{13-6(z+z^{-1})}$

$$= -\frac{i}{2}\cdot\frac{z^2+z^{-2}}{13z-6z^2-6}$$

$6z^2-13z+6 = (3z-2)(2z-3)$

$$= \frac{i}{2}\cdot\frac{z^4+1}{z^2(6z^2-13z+6)} = \frac{i}{6}\cdot\frac{z^4+1}{z^2(z-\frac{2}{3})(2z-3)}$$

inside $|z|=1$

$$\operatorname*{Res}_{z=0} f(z) = \frac{i}{2}\cdot\frac{d}{dz}\left(\frac{z^4+1}{6z^2-13z+6}\right)_{z=0}$$

$$= \frac{i}{2}\cdot\frac{4z^3(6z^2-13z+6)-(z^4+1)(12z-13)}{(6z^2-13z+6)^2}\Bigg|_{z=0}$$

$$= \frac{i}{2}\cdot\frac{13}{36} = \frac{13i}{72}$$

$$\operatorname*{Res}_{z=\frac{2}{3}} f(z) = \frac{i}{6}\cdot\frac{z^4+1}{z^2(2z-3)}\Bigg|_{z=\frac{2}{3}}$$

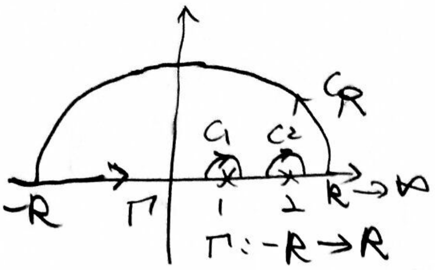
$$= \frac{i}{6}\cdot\frac{\frac{16}{81}+1}{\frac{4}{9}\cdot(\frac{4}{3}-3)} = -\frac{i}{6}\cdot\frac{\cancel{\frac{97}{81}}^{3}}{\cancel{\frac{20}{27}}_{1}} = -\frac{97i}{360}$$

$$\Rightarrow I = 2\pi i\cdot\left(\frac{13i}{72} - \frac{97i}{360}\right)$$

$$= 2\pi i\cdot\frac{65-97}{360}\,i = 2\pi\,\frac{\cancel{32}^{4}}{\cancel{360}_{45}} = \frac{8\pi}{45}$$ ※

9 (b). Find P.V. $\int_{-\infty}^{\infty} \frac{x\cos x}{x^2-3x+2}\,dx$.

Let $f(z)=\frac{ze^{iz}}{z^2-3z+2}=\frac{z\cdot e^{iz}}{(z-1)(z-2)}$



$C=C_R+\Gamma+C_1+C_2$

$$\oint_C f(z)\,dz = \int_{C_R+\Gamma+C_1+C_2} f(z)\,dz.$$
$$=0$$

where $\int_{C_R} f(z)\,dz=0$ as $R\to\infty$
(proof is omitted!)

$$\int_{\Gamma_1} f(z)\,dz = \text{p.v.} \int_{-\infty}^{\infty}\frac{xe^{ix}}{x^2-3x+2}\,dx$$

$$\int_{C_1} f(z)\,dz = -\pi i\cdot \operatorname*{Res}_{z=1} f(z) = -\pi i\cdot\left(\frac{ze^{iz}}{z-2}\right)_{z=1}$$
$$= -\pi i\cdot(-e^{i}) = \pi i(\cos 1 + i\sin 1)$$
$$= -\pi\sin 1 + i\cdot\pi\cos 1.$$

$$\int_{C_2} f(z)\,dz = -\pi i\cdot \operatorname*{Res}_{z=2} f(z) = -\pi i\left(\frac{ze^{iz}}{z-1}\right)_{z=2}$$
$$= -\pi i(2e^{i2}) = -2\pi i(\cos 2 + i\sin 2)$$
$$= 2\pi\sin 2 - i\cdot 2\pi\cos 2$$

$$\Rightarrow \text{p.v.}\int_{-\infty}^{\infty}\frac{x\cos x}{x^2-3x+2}\,dx + i\cdot\text{p.v.}\int_{-\infty}^{\infty}\frac{x\sin x}{x^2-3x+2}\,dx$$
$$=\pi(\sin 1-2\sin 2)+i\pi(2\cos 2-\cos 1)$$

$$\therefore\ \text{p.v.}\int_{-\infty}^{\infty}\frac{x\cos x}{x^2-3x+2}\,dx = \pi(\sin 1 - 2\sin 2)$$ #