Complex Analysis, 2016 Midterm DCChang. 2016/11/23 ①

\#1 Let $z = x+iy$, $x, y \in R$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$
$$= \frac{1}{2i}(e^{ix-y} - e^{-ix+y})$$
$$= \frac{1}{2i}[e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)]$$
$$= \frac{1}{2}(e^{-y}+e^{y})\sin x - \frac{i}{2}(e^{-y}-e^{y})\cos x$$
$$= 2$$

$$\Rightarrow \begin{cases}(e^{-y}+e^{y})\sin x = 4 & \text{— ①}\\ (e^{-y}-e^{y})\cos x = 0 & \text{— ②}\end{cases}$$

From ②: $e^{-y} = e^{y}$ or $\cos x = 0$

1°. if $e^{-y} = e^{y}$, $(e^{y})^2 = 1 \Rightarrow e^{y} = \pm 1$

but from ①, this leads to $\sin x = \pm 2$ (Not Valid for $x \in R$)

2°. if $\cos x = 0 \Rightarrow x = \frac{\pi}{2} + 2n\pi$ or $x = \frac{3\pi}{2} + 2n\pi$, $n = 0, \pm1, \pm2, \cdots$

$\Rightarrow \sin x = 1$ or $\sin x = -1$

Case (a): $\sin x = 1$	case (b): $\sin x = -1$
$\Rightarrow e^{-y} + e^{y} = 4$	$\Rightarrow e^{-y} + e^{y} = -4$
$(e^{y})^2 - 4e^{y} + 1 = 0$	$(e^{y})^2 + 4e^{y} + 1 = 0$
$\Rightarrow e^{y} = 2 \pm \sqrt{3}$	$\Rightarrow e^{y} = -2 \pm \sqrt{3} < 0$
$y = \ln(2 \pm \sqrt{3})$	Since y is real, this is not valid.

From 1° and 2°, we have

$$z = \left(\frac{1}{2} + 2n\right)\pi + i\ln(2 \pm \sqrt{3}),\ n = 0, \pm1, \pm2, \cdots$$

or $z = \left(\frac{1}{2} + 2n\right)\pi - i\ln(2 \pm \sqrt{3})$ $\because (2\pm\sqrt{3})^{-1} = 2 \mp \sqrt{3}$ \#

#2

(a) To verify that u is harmonic, we have to show that $u_{xx}+u_{yy}=0$.

$u_x = 3x^2-3y^2$, $u_{xx}=6x$

$u_y=-6xy-5$, $u_{yy}=-6x$

$\Rightarrow u_{xx}+u_{yy}=0$. $\therefore u(x,y)$ is harmonic #

(b) Let v be a harmonic conjugate of u, then $f(x,y)=u+iv$ is analytic. So, we have

$$u_x=v_y \text{ and } u_y=-v_x$$

$\Rightarrow v_y=3x^2-3y^2 \Rightarrow v=3x^2y-y^3+h(x)$

and, $v_x=6xy+h'(x)=-u_y$
$=6xy+5$

$\Rightarrow h'(x)=5$ i.e. $h(x)=5x+C$, C: const.

Finally, we have $\underline{v(x,y)=3x^2y-y^3+5x+C}$ #

#3.

$$\oint_C \frac{z+1}{z^5}\cdot f(z)\,dz=\oint_C \frac{z+1}{z^5}\cdot\sum_{n=0}^{\infty}\frac{n^4}{4^n}z^n\,dz$$

$$=\oint_C\sum_{n=0}^{\infty}\frac{n^4}{4^n}(z+1)z^{n-5}\,dz$$

$$=\cancel{\cdots}^{\,0}+\cancel{\oint_C\frac{5^4}{4^5}(z+1)z^0\,dz}^{\,0}+\oint_C\frac{4^4}{4^4}(z+1)\cdot z^{-1}dz$$

$$+\oint_C\frac{3^4}{4^3}(z+1)\cdot z^{-2}dz+\oint_C\frac{2^4}{4^2}(z+1)z^{-3}dz+\cdots$$

$$=\oint_C\frac{z+1}{z}dz+\frac{81}{64}\oint_C\frac{z+1}{z^2}dz+\oint_C\frac{z+1}{z^3}dz+\cdots$$

$$=2\pi i\,(z+1)\big|_{z=0}+\frac{81}{64}\cdot 2\pi i\cdot(z+1)'\big|_{z=0}+\cancel{2\pi i\,(z+1)''\big|_{z=0}}^{\,0}+\cancel{\cdots}^{\,0}$$

$$=2\pi i+\frac{81}{32}\pi i=\underline{\frac{145}{32}\pi i}$$ #

#4.

(a) $\lim_{n\to\infty}\left|\frac{i^{n+1}/(1+i)^n}{i^n/(1+i)^{n-1}}\right| = \lim_{n\to\infty}\left|\frac{i}{1+i}\right| = \frac{1}{\sqrt{2}} < 1$ $\therefore$ converge! #

(b) $\lim_{n\to\infty}\left|\frac{(1+i)^{n+1}/(-1)^{n+1}(n+1)^3}{(1+i)^n/(-1)^n n^3}\right| = \lim_{n\to\infty}\left|\frac{(1+i)\cdot n^3}{(-1)(n+1)^3}\right|$

$= \lim_{n\to\infty}\sqrt{2}\cdot\left(\frac{1}{1+\frac{1}{n}}\right)^3 = \sqrt{2} > 1$ $\therefore$ diverge! #

(c)

$\because \sum_{n=1}^{\infty}\frac{i^n}{n^2} \le \sum_{n=1}^{\infty}\left|\frac{i^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent, so $\sum_{n=1}^{\infty}\frac{i^n}{n^2}$ Converges! #

(d) By ratio test, $\lim_{n\to\infty}\left|\frac{i^{n+1}/2}{i^n/2}\right| = 1$, inconclusive for convergence.

However, $\because \lim_{n\to\infty}\left(\frac{i^n}{2}\right) \neq 0$ $\therefore \sum_{n=0}^{\infty}\frac{i^n}{2}$ diverges! #

(e) $\lim_{n\to\infty}\left|\frac{[(n+1)i]^{n+1}/(n+1)!}{(ni)^n/n!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}\cdot i}{n^n\cdot(n+1)}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^n}{n^n}\right|$

$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$ $\therefore$ diverge! #

#5.

(a) $R = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}/n!}{(-1)^{n+2}/(n+1)!}\right| = \lim_{n\to\infty}(n+1) = \infty$ $\therefore$ radius $= \infty$ #

(b) $R = \lim_{n\to\infty}\left|\frac{\left(\frac{6n+1}{2n+5}\right)^n}{\left(\frac{6(n+1)}{2(n+1)+5}\right)^{n+1}}\right| = \lim_{n\to\infty}\left(\frac{\left(\frac{6}{2}\right)^n}{\left(\frac{6}{2}\right)^{n+1}}\right) = \frac{1}{3}$

$\therefore$ radius $= \frac{1}{3}$ #

(c) By ratio test

$\lim_{n\to\infty}\left|\frac{z^{2(n+1)}/4^{n+1}}{z^{2n}/4^n}\right| < 1$ $\therefore \lim_{n\to\infty}\left|\frac{z^2}{4}\right| < 1$

$\Rightarrow |z^2| < 4$, $|z| < 2$ $\therefore$ radius $= 2$ #

#6. $0<|z|<1$

④

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n$$

(a) $\because |z^2|<1$

$$\therefore f(z)=\frac{1}{1+z^2}=\frac{1}{1-(-z^2)}=\sum_{n=0}^{\infty}(-z^2)^n=\sum_{n=0}^{\infty}(-1)^n z^{2n}$$

$$=\underline{1-z^2+z^4-z^6+\cdots}\ \#$$

(b)

$$\frac{d}{dz}\left(\frac{1}{1-z}\right)=\frac{1}{(1-z)^2}=\frac{d}{dz}\left(\sum_{n=0}^{\infty}z^n\right)=\sum_{n=0}^{\infty}(n+1)z^n$$

$$\therefore f(z)=\frac{1}{(1+z)^2}=\frac{1}{[1-(-z)]^2}=\sum_{n=0}^{\infty}(n+1)(-z)^n$$

$$=\underline{1-2z+3z^2-4z^3+\cdots}\ \#$$

or by Taylor's expansion,

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n$$

$$f^{(0)}(0)=1\ ,\ f^{(1)}(0)=-\left.\frac{2}{(1+z)^3}\right|_{z=0}=-2$$

$$f^{(2)}(0)=\left.\frac{3!}{(1+z)^4}\right|_{z=0}=6$$

$$f^{(3)}(0)=\left.\frac{-4!}{(1+z)^5}\right|_{z=0}=-24$$

$$\vdots$$

$$\therefore f(z)=\sum_{n=0}^{\infty}(-1)^n\frac{(n+1)!}{n!}z^n=\sum_{n=0}^{\infty}(-1)^n(n+1)z^n$$

$$=1-2z+3z^2-4z^3+\cdots$$

#7.

(a) $f(z) = \frac{1}{(z-1)^2} \cdot \frac{1}{z-3}$

$= \frac{1}{(z-1)^2} \cdot \frac{1}{(z-1)-2}$

$= \frac{1}{(z-1)^2} \cdot \left(-\frac{1}{2}\right) \cdot \frac{1}{1-\left(\frac{z-1}{2}\right)}$ $\qquad \because |z-1| < 2$

$= \left(-\frac{1}{2}\right) \cdot \frac{1}{(z-1)^2} \cdot \sum_{n=0}^{\infty} \left(\frac{z-1}{2}\right)^n$ $\qquad \therefore \left|\frac{z-1}{2}\right| < 1$

$= -\sum_{n=0}^{\infty} \frac{(z-1)^{n-2}}{2^{n+1}}$

$= -\frac{1}{2(z-1)^2} - \frac{1}{4(z-1)} - \frac{1}{8} - \frac{1}{16}(z-1) - \cdots$ #

(b) $f(z) = \frac{1}{z-3} \cdot \frac{1}{(z-1)^2}$

$= \frac{1}{z-3} \cdot \frac{1}{[(z-3)+2]^2}$

$= \frac{1}{z-3} \cdot \frac{1}{(z-3)^2} \cdot \frac{1}{\left(1+\frac{2}{z-3}\right)^2}$ $\qquad \because |z-3| > 3$

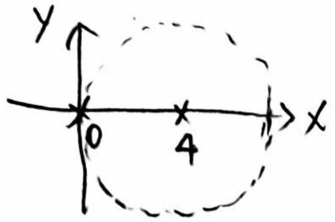
$\qquad \therefore \left|\frac{2}{z-3}\right| < 1.$

$= \frac{1}{(z-3)^3} \cdot \sum_{n=0}^{\infty} (-1)^n \cdot (n+1) \left(\frac{2}{z-3}\right)^n$

$= \sum_{n=0}^{\infty} (-1)^n (n+1) 2^n \cdot (z-3)^{-(n+3)}$

$= \frac{1}{(z-3)^3} - \frac{4}{(z-3)^4} + \frac{12}{(z-3)^5} - \frac{32}{(z-3)^6} + \cdots$ #

#8. $f(z) = \dfrac{z+1}{z(z-4)^3}$



$f(z)$ has two poles at $z=0$ and 4

We consider the domains for expansion

(a) $0 < |z-4| < 4$ (b) $|z-4| > 4$

(a) for $0 < |z-4| < 4$

$$f(z) = \frac{1}{(z-4)^3} \cdot \frac{z+1}{z}$$

$$= \frac{1}{(z-4)^3}\left(1 + \frac{1}{z}\right)$$

$$= \frac{1}{(z-4)^3}\left[1 + \frac{1}{(z-4)+4}\right]$$

$$= \frac{1}{(z-4)^3}\left[1 + \frac{1}{4}\,\frac{1}{1+\left(\frac{z-4}{4}\right)}\right]$$

$\because \left|\frac{z-4}{4}\right| < 1$

$$= \frac{1}{(z-4)^3}\left[1 + \frac{1}{4}\cdot\sum_{n=0}^{\infty}\left(-\frac{z-4}{4}\right)^n\right]$$

$$= \frac{1}{(z-4)^3} + \sum_{n=0}^{\infty}(-1)^n \cdot \frac{(z-4)^{n-3}}{4^{n+1}}$$

$$= \frac{\frac{5}{4}}{(z-4)^3} - \frac{\frac{1}{16}}{(z-4)^2} + \frac{\frac{1}{64}}{(z-4)} - \frac{1}{256} + \cdots$$

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(b) for $|z-4| > 4$

$$f(z) = \frac{1}{(z-4)^3}\cdot\left[1 + \frac{1}{z-4}\cdot\frac{1}{1+\left(\frac{4}{z-4}\right)}\right]$$

$\because \left|\frac{4}{z-4}\right| < 1$

$$= \frac{1}{(z-4)^3}\left[1 + \frac{1}{z-4}\cdot\sum_{n=0}^{\infty}\left(-\frac{4}{z-4}\right)^n\right]$$

$$= \frac{1}{(z-4)^3} + \sum_{n=0}^{\infty}(-1)^n\frac{4^n}{(z-4)^{n+4}}$$

$$= \frac{1}{(z-4)^3} + \frac{1}{(z-4)^4} - \frac{4}{(z-4)^5} + \frac{16}{(z-4)^6} - \cdots$$

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