Complex Anlysis Quiz-1 Solution

Oct. 28, 2016 DCChang

1.

$$\sin z_2 - \sin z_1 = (e^{iz_2} - e^{-iz_2} - e^{iz_1} + e^{-iz_1})/2i$$

$$= 2[e^{i(z_2+z_1)/2} + e^{-i(z_2+z_1)/2})/2][e^{i(z_2-z_1)/2} - e^{-i(z_2-z_1)/2})/2i]$$

$$= 2\cos((z_2+z_1)/2)\sin((z_2-z_1)/2)$$

$$= 0 \text{ if } (z_2 + z_1)/2 = \pi/2 + k\pi \text{ or } (z_2 - z_1)/2 = k\pi,$$

or if $z_2 = -z_1 + (2k+1)\pi$ or $z_2 = z_1 + 2k\pi$ where k is an integer.

2.

(a)

$$z^2 - 1 = e^{\mathrm{Log}(z^2-1)} = e^{i\pi/2} = i$$

$$\Longleftrightarrow z^2 = 1 + i$$

$$\Longleftrightarrow z = (1+i)^{1/2} = \sqrt[4]{2}e^{i\pi/8},\ \sqrt[4]{2}e^{i9\pi/8}$$

(b)

(First use the quadratic formula)

$$e^z = \frac{-1+\sqrt{-3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\begin{aligned} z &= \log\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) \\ &= \mathrm{Log}\left|-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right| + i\arg\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) \\ &= i\left(\pm\frac{2\pi}{3} + 2k\pi\right),\quad k = 0, \pm 1, \ldots \end{aligned}$$

3.

(a)

$$\begin{aligned} (1+i)^{1-i} &= (1+i)(1+i)^{-i} \\ &= (1+i)e^{-i\log(1+i)} \\ &= (1+i)\exp\left[\frac{\pi}{4} + 2k\pi - \frac{i}{2}\mathrm{Log}\,2\right],\quad k = 0, \pm 1, \ldots \end{aligned}$$

(b) $(1+i)^{(1+i)} = (1+i)e^{i\mathrm{Log}(1+i)} = (1+i)\exp\left[-\frac{\pi}{4} + \frac{i}{2}\mathrm{Log}\,2\right]$

4.

(i) Prove continuity: (given a epsilon>0 for |g(z)-g(z_0)|<epsilon, we can find a delta such that |z-z_0|<delta with delta=epsilon, satisfying the mathematical definition of continuity),

Let z_0 be any complex number. Given $\varepsilon > 0$ choose $\delta = \varepsilon$. Then whenever $|z - z_0| < \delta$,

$$|g(z) - g(z_0)| = |\overline{z} - \overline{z}_0| = |\overline{z - z_0}| = |z - z_0| < \varepsilon.$$

(ii) Prove nowhere differentiable: (Not satisfying Cauchy-Riemnan equations)

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$$

5.

5. $\dfrac{\partial u}{\partial x} = 2e^{x^2-y^2}[x\cos(2xy) - y\sin(2xy)] = \dfrac{\partial v}{\partial y}$

$\dfrac{\partial u}{\partial y} = -2e^{x^2-y^2}[y\cos(2xy) + x\sin(2xy)] = -\dfrac{\partial v}{\partial x}$

f is entire because these first partials exist and are continuous for all x and y.

$$\begin{aligned} f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} &= 2e^{x^2-y^2}(x+iy)[\cos(2xy) + i\sin(2xy)] \\ &= 2e^{(x^2-y^2)}e^{i2xy}(x+iy) \\ &= 2ze^{z^2} \end{aligned}$$

6.

$\dfrac{\partial u}{\partial x} = \cos x \cosh y = \dfrac{\partial v}{\partial y}$

$\Rightarrow v(x,y) = \int \cos x \cosh y\, dy = \cos x \sinh y + \psi(x)$

$\dfrac{\partial u}{\partial y} = \sin x \sinh y = -\dfrac{\partial v}{\partial x} = \sin x \sinh y + \psi'(x) \Rightarrow \psi(x) = a$

Thus, $v(x,y) = \cos x \sinh y + a$.

7.

$$1+z+z^2+\cdots+z^n=\frac{z^{n+1}-1}{z-1} \quad \text{when} \quad z\neq 1$$

Suppose $z=e^{i\theta}$, $\theta\neq 0$. Then

$$\begin{aligned}1+z+z^2+\cdots+z^n &= 1+e^{i\theta}+e^{i2\theta}+\cdots+e^{in\theta}\\ &= (1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta)\\ &\quad +i(\sin\theta+\sin 2\theta+\cdots+\sin n\theta)\end{aligned}$$

and

$$\begin{aligned}\frac{z^{n+1}-1}{z-1} &= \frac{e^{i(n+1)\theta}-1}{e^{i\theta}-1}\\ &= \frac{\cos(n+1)\theta-1+i\sin(n+1)\theta}{(\cos\theta-1)^2+\sin^2\theta}(\cos\theta-1-i\sin\theta)\\ &= \frac{\cos n\theta-\cos(n+1)\theta-\cos\theta+1}{2-2\cos\theta}\\ &\quad -i\frac{\sin n\theta-\sin(n+1)\theta+\sin\theta}{2-2\cos\theta}\end{aligned}$$

Note that in the real part,

$$1-\cos\theta=2sin^2(\frac{\theta}{2})$$

$$\begin{aligned}\cos(n\theta)-\cos(n+1)\theta &= \cos\left[\left(n+\tfrac{1}{2}\right)\theta-\tfrac{\theta}{2}\right]-\cos\left[\left(n+\tfrac{1}{2}\right)\theta+\tfrac{\theta}{2}\right]\\ &= \cos\left(n\theta+\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)+\sin\left(n\theta+\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\\ &\quad -\left[\cos\left(n\theta+\tfrac{\theta}{2}\right)\cos\left(\tfrac{\theta}{2}\right)-\sin\left(n\theta+\tfrac{\theta}{2}\right)\sin(\tfrac{\theta}{2})\right]\\ &= 2\sin\left[\left(n+\tfrac{1}{2}\right)\theta\right]\sin\left(\tfrac{\theta}{2}\right) \text{ ....*註}\end{aligned}$$

(*註: 未寫出如此式說明原因而立刻寫出下式結果者, 解答不完整, 本題扣 10 分)

$$\frac{\cos n\theta-\cos(n+1)\theta-\cos\theta+1}{2-2\cos\theta}=\frac{\sin(n+1/2)\theta+\sin\theta/2}{2\sin\theta/2}$$

⇨ $$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta=\frac{\sin(n+1/2)\theta+\sin\theta/2}{2\sin\theta/2}$$

$$=\frac{1}{2}+\frac{\sin[(n+\frac{1}{2})\theta]}{2\sin(\theta/2)}.$$