#1. $f(z)=3(z-i)^{-2}+2(z-i)^{-1}+1-2(z-i)-3(z-i)^{3}$

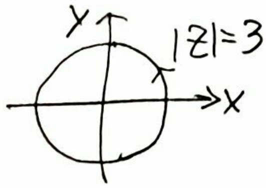
$C: |z-i|=2 \qquad \because z_0=i$ inside C.

$\oint_C f(z)dz = \oint_C \frac{2}{z-i}dz = -2\pi i \times 2$

$= -4\pi i$ #

#2 (a) $z=3e^{i\theta}$, $0\le\theta<2\pi$ for counterclockwise traverse.

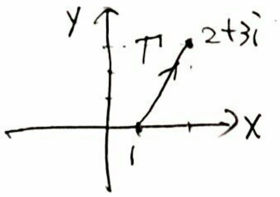
$\bar{z}=3e^{-i\theta}$, $dz=3ie^{i\theta}d\theta$



$\oint_C \bar{z}\,dz = \int_0^{2\pi} 3e^{-i\theta}\cdot 3ie^{i\theta}d\theta$

$= 9i\int_0^{2\pi}d\theta = 18\pi i$ #

(b)



$\Gamma: z(t)=(1+(2-1)t)+(3-0)t\,i$

$=(1+t)+i\cdot 3t, \quad 0\le t\le 1$

$dz=(1+3i)dt$, $Re(z)=1+t$

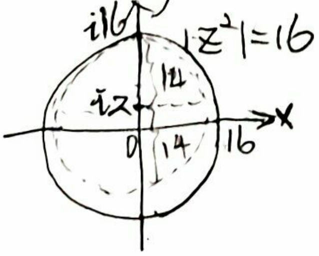
$\int_\Gamma Re(z)dz = \int_0^1 (1+t)\cdot(1+3i)dt$

$=(1+3i)\int_0^1(1+t)dt$

$=(1+3i)\cdot\left(t+\frac{1}{2}t^2\right)\Big|_0^1$

$=\frac{3}{2}+i\frac{9}{2}$ #

#3.



$C: |z|=4$, $z=4e^{i\theta}$ $\therefore |z^2|=16$

$|z^2-2i|\ge 14$, By ML-inequality,

$\left|\oint_C \frac{dz}{z^2-2i}\right| \le \oint_C \left|\frac{1}{z^2-2i}\right||dz| \le \frac{1}{14}\oint_C dz = \frac{4\cdot 2\pi}{14} = \frac{4\pi}{7}$ #

RZ.

#4 $\because f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$

$C: |z|=2$, $z_0 = -i$ inside C , $f(z) = 5z^3+2z+1$

$$\oint_C \frac{5z^3+2z+1}{(z+i)^3}\,dz = \frac{2\pi i}{2!} f^{(2)}(-i)$$
$$= \pi i\,(5z^3+2z+1)''\Big|_{z=-i}$$
$$= \pi i\cdot 30z\Big|_{z=-i} = 30\pi \;\#$$

#5.

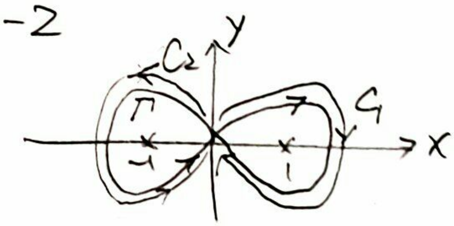
$$f(z) = \frac{2z^2-z+1}{z^3+z^2-z-1} = \frac{2z^2-z+1}{(z-1)(z+1)^2} = \frac{A}{z-1} + \frac{B}{z+1} + \frac{C}{(z+1)^2}$$

$$A = \lim_{z\to 1}(z-1)f(z) = \frac{2z^2-z+1}{(z+1)^2}\Big|_{z=1} = \frac{2}{4} = \frac{1}{2}$$

$$B = \lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2 f(z)\right] = \left(\frac{2z^2-z+1}{z-1}\right)'\Big|_{z=-1} = \frac{3}{2}$$

$$C = \lim_{z\to -1}(z+1)^2 f(z) = \frac{2z^2-z+1}{z-1}\Big|_{z=-1} = \frac{2+1+1}{-2} = -2$$

$$f(z) = \frac{\frac{1}{2}}{z-1} + \frac{\frac{3}{2}}{z+1} + \frac{-2}{(z+1)^2}$$



$$\int_\Gamma f(z)\,dz = \oint_{C_1}\frac{\frac{1}{2}}{z-1}\,dz + \oint_{C_2}\frac{\frac{3}{2}}{z+1}\,dz \qquad \Gamma = C_1 + C_2$$
$$= -2\pi i\cdot\left(\frac{1}{2}\right) + 2\pi i\cdot\left(\frac{3}{2}\right) = 2\pi i \;\#$$

#6. $f(z) = \frac{z+i}{z^3+2z^2+z} = \frac{z+i}{z(z+1)^2}$

(a) $\oint_{|z|=1} f(z)\,dz = \oint_{|z|=1}\frac{(z+i)/(z+1)^2}{z}\,dz = 2\pi i\left[\frac{z+i}{(z+1)^2}\right]_{z=0}$

$= -2\pi \;\#$

(b) $\oint_{|z+2-i|=2} f(z)\,dz = \oint \frac{(z+i)/z}{(z+1)^2}\,dz = -2\pi i\left(\frac{z+i}{z}\right)'\Big|_{z=-1}$

$= -2\pi i\cdot\frac{z-(z+i)}{z^2}\Big|_{z=-1} = -2\pi i\cdot(-i) = -2\pi \;\#$

#7

$\because f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$

$C: |z-z_0| = r$, Let $z = z_0 + re^{i\theta}$, $0 \le \theta < 2\pi$

$dz = ire^{i\theta} d\theta$

$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{(re^{i\theta})^{n+1}} \cdot ire^{i\theta} d\theta$

$= \frac{n!}{2\pi r^n} \int_0^{2\pi} f(z_0+re^{i\theta}) \cdot e^{-in\theta} d\theta$ #

#8 (a) $f^{(n)}(0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz$

$|f^{(n)}(0)| \le \frac{n!}{2\pi} \oint_C \left| \frac{f(z)}{z^{n+1}} \right| dz$

Let $C: |z| = R$ $\because f(z)$ is analytic for $|z|<1$.

$|f(z)| < \frac{1}{1-|z|} = \frac{1}{1-R}$

$\therefore |f^{(n)}(0)| \le \frac{n!}{2\pi} \oint_C \frac{1}{R^{n+1}} \cdot \frac{1}{1-R} dz$

$= \frac{n!}{\cancel{2\pi}} \cdot \frac{1}{R^{n+1}} \cdot \frac{1}{1-R} \cdot \cancel{2\pi} R$

$= \frac{n!}{R^n(1-R)}$ #

(b) upper bound is to find minimum R.

$\frac{d}{dR} \frac{n!}{R^n(1-R)} = 0 \Rightarrow \frac{n!R^n - n! \cdot nR^{n-1}(1-R)}{R^{2n}(1-R)^2} = 0.$

$\therefore n! \cdot R^{n-1}[(n+1)R - n] = 0$.

$\Rightarrow R = \frac{n}{n+1}$ #