Complex Analysis Quiz-3 2016/12/19 D.C. Chang NCHU/CE. ①

#1. The pole of $f(z)$ is to find z such that

$$g(z) = h^2(z) = (6\sin z - 1 + z^3)^2 = 0, \quad h(z) = 6\sin z - 1 + z^3$$

Roughly observing $h(z)$, we see that the zero of $h(z)$ is within $0 < z < \frac{\pi}{3}$. Using the approximation $\sin z \approx z - \frac{z^3}{6}$,

$$g(z) = h^2(z) \approx (6z - z^3 - 1 + z^3)^2 = (6z - 1)^2$$

$g(z)$ has a zero of order 2 around $z = \frac{1}{6}$

$\Rightarrow$ $f(z)$ has a pole of order 2 (around $z = \frac{1}{6}$) ※

#2. $f(z)$ has a zero of order m at $z = z_0$

$f(z) = (z - z_0)^m \cdot g(z)$ $\quad g(z)$ is analytic at z_0.

$$\frac{f'(z)}{f(z)} = \frac{m(z-z_0)^{m-1} g(z) + (z-z_0)^m g'(z)}{(z-z_0)^m g(z)}$$

$$= \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}, \quad \frac{g'(z)}{g(z)} \text{ is analytic at } z_0.$$

$$\oint_C \frac{f'(z)}{f(z)} dz = \oint_C \frac{m}{z - z_0} dz + \underbrace{\oint_C \frac{g'(z)}{g(z)} dz}_{\to 0} = i\,2\pi m$$ ※

#3.

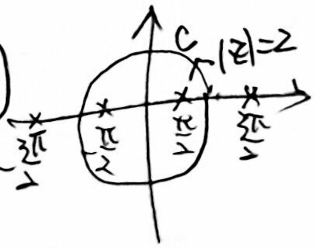
(a) $\underset{z=z_0}{\operatorname{Res}} \tan z = \underset{z=z_0}{\operatorname{Res}} \frac{\sin z}{\cos z}$ $\qquad z_0$: singular points.

$\because \cos z$ has zeros at $z = \frac{\pi}{2} + n\pi$, $n = 0, \pm 1, \pm 2, \cdots$

which are all simple poles.

$$\underset{z=z_0}{\operatorname{Res}} \frac{\sin z}{\cos z} = \left.\frac{\sin z}{(\cos z)'}\right|_{z=\frac{\pi}{2}+n\pi} = -1 \quad \text{for } z_0 = \frac{\pi}{2} + n\pi, \ n = 0, \pm 1, \pm 2, \cdots$$

$$\oint_{|z|=2} \tan z\, dz = 2\pi i \cdot \left(\underset{z=\frac{\pi}{2}}{\operatorname{Res}} \tan z + \underset{z=-\frac{\pi}{2}}{\operatorname{Res}} \tan z\right)$$



$$= 2\pi i \cdot [-1 + (-1)]$$

$$= -4\pi i$$ ※

(b) $\frac{1}{z\sin z}$ has poles at $z=0, \pm\pi, \pm 2\pi, \cdots$

Since $C: |z|=1$, we consider the residue at $z=0$

$\because$ $z\sin z$ has a zero of order 2 at $z=0$.

$$\operatorname*{Res}_{z=0} \frac{1}{z\sin z} = \lim_{z\to 0} \frac{d}{dz}\left(z^2 \cdot \frac{1}{z\sin z}\right)$$

$$= \lim_{z\to 0} \frac{d}{dz}\left(\frac{z}{\sin z}\right)$$

$$= \lim_{z\to 0} \frac{\sin z - z\cos z}{\sin^2 z}$$

L'Hopital's rule.

$$= \left.\frac{\cos z - (\cos z - z\sin z)}{2\cos z\sin z}\right|_{z=0}$$

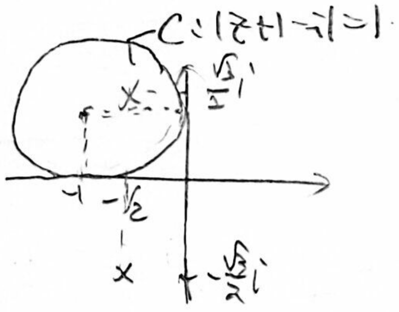
$$= \left.\frac{z}{2\cos z}\right|_{z=0} = 0$$

$$\oint_{|z|=1} \frac{1}{z\sin z}\,dz = 0 \ \#$$

(c)

$$f(z) = \frac{1}{z^2+z+1} = \frac{1}{(z-z_0)(z-z_1)}$$

$$z_0 = \frac{-1+\sqrt{3}i}{2}, \quad z_1 = \frac{-1-\sqrt{3}i}{2}$$



$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{z_0-z_1} = \frac{1}{\sqrt{3}i}$$

$$\oint_C f(z)\,dz = 2\pi i \cdot \operatorname*{Res}_{z=z_0} f(z) = \frac{2\pi}{\sqrt{3}} \ \#$$

(d)

$$e^{iz} = 1 + iz + \frac{(iz)^2}{2} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \cdots$$

$$= 1 + iz - \frac{1}{2}z^2 - \frac{z^3}{3!}i + \frac{z^4}{4!} + \cdots$$

$$= \left(1 - \frac{1}{2}z^2 + \frac{z^4}{4!}\cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)$$

$$= \cos z + i\sin z \quad \text{for } |z|<1.$$

$$\therefore \sin\left(\frac{1}{2z}\right) = \frac{1}{2z} - \frac{1}{6}\left(\frac{1}{2z}\right)^3 + \frac{1}{5!}\left(\frac{1}{2z}\right)^5 - \cdots$$

$$\operatorname*{Res}_{z=0} z^2\sin\left(\frac{1}{2z}\right) = -\frac{1}{6}\times\frac{1}{8} = -\frac{1}{48}$$

$$\Rightarrow \oint_{|z|=1} z^2\sin\left(\frac{1}{2z}\right)dz = 2\pi i \cdot \operatorname*{Res}_{z=0} z^2\sin\left(\frac{1}{2z}\right) = -\frac{\pi i}{24} \ \#$$

#4(a)

$\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$ $\qquad z=e^{i\theta},\ dz = ie^{i\theta}d\theta = iz\,d\theta$

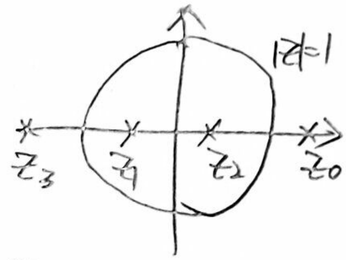
$\sin^2\theta = -\frac{1}{4}\left(z^2+\frac{1}{z^2}-2\right)$ $\qquad C: z=e^{i\theta},\ -\pi\le\theta\le\pi$

$$\frac{1}{1+\sin^2\theta} = \frac{1}{1-\frac{z^4-2z^2+1}{4z^2}} = \frac{-4z^2}{z^4-6z^2+1}$$

$$\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta} = \oint_C \frac{-4z^2}{z^4-6z^2+1}\cdot\frac{1}{iz}dz = 4i\oint_C \frac{z}{z^4-6z^2+1}dz$$

$$f(z) = \frac{z}{z^4-6z^2+1} = \frac{z}{(z^2-2z-1)(z^2+2z-1)}$$

$$= \frac{z}{(z-z_0)(z-z_1)(z-z_2)(z-z_3)}$$



$z_0, z_1 = 1\pm\sqrt{2}$, $z_2, z_3 = -1\pm\sqrt{2}$

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{z_1}{(z_1-z_0)(z_1-z_2)(z_1-z_3)} = \frac{1-\sqrt{2}}{-2\sqrt{2}\cdot(2-2\sqrt{2})\cdot 2} = -\frac{1}{8\sqrt{2}}$$

$$\operatorname*{Res}_{z=z_2} f(z) = \frac{z_2}{(z_2-z_0)(z_2-z_1)(z_2-z_3)} = \frac{-1+\sqrt{2}}{-2\cdot(-2+2\sqrt{2})\cdot 2\sqrt{2}} = -\frac{1}{8\sqrt{2}}$$

$$\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta} = 4i\cdot 2\pi i\cdot\left(-\frac{2}{8\sqrt{2}}\right) = \sqrt{2}\pi$$ #

(b)

$$\int_0^{2\pi} e^{\cos\theta}\cos(n\theta-\sin\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} e^{\cos\theta}\left[e^{i(n\theta-\sin\theta)} + e^{-i(n\theta-\sin\theta)}\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(e^{in\theta}\cdot e^{\cos\theta - i\sin\theta} + e^{-in\theta}\cdot e^{\cos\theta+i\sin\theta}\right)d\theta$$

$$= \frac{1}{2}\oint_{|z|=1}\left(z^n e^{\frac{1}{z}} + z^{-n}\cdot e^{z}\right)\cdot\frac{1}{iz}dz$$

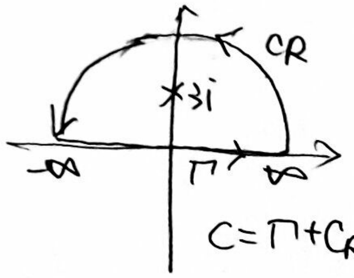
$$= \frac{1}{2i}\left(\oint_{|z|=1} z^{n-1}\cdot e^{\frac{1}{z}}dz + \oint_{|z|=1} z^{-(n+1)}\cdot e^{z}dz\right)$$

$$= \frac{1}{2i}\Big(\oint_{|z|=1} z^{n-1}\left(1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3!\,z^3}+\cdots\right)dz$$

$$+\oint_{|z|=1} z^{-(n+1)}\cdot\left(1+z+\frac{z^2}{2}+\frac{z^3}{3!}+\cdots\right)dz$$

$$= \frac{1}{2i}\cdot 2\pi i\cdot\left(\frac{1}{n!}+\frac{1}{n!}\right) = \frac{2\pi}{n!}$$ #

#4(c) Let $f(z) = \dfrac{\frac{1}{2}(e^{i2z}+e^{-i2z})}{z-3i}$



$\oint_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-\infty}^{\infty} f(x)\,dx$.

$\because \int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$.

$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \oint_C f(z)\,dz$.

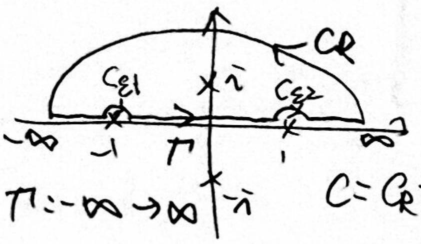
$$\int_{-\infty}^{\infty} \frac{\cos 2x}{x-3i}\,dx = \frac{1}{2}\oint_C \frac{e^{i2z}}{z-3i}\,dz + \frac{1}{2}\oint_C \frac{e^{-i2z}}{z-3i}\,dz$$

$$= \frac{2\pi i}{2}\operatorname*{Res}_{z=3i} e^{i2z} + \frac{1}{2}\cdot \underbrace{\int_{-\Gamma} \frac{e^{i2z'}}{z'+3i}\,dz'}_{\text{pole is outside } C} \nearrow 0 \qquad z' = -z$$

$$= \pi i e^{-6}$$ #

#4(d) $\displaystyle\int_0^{\infty} \frac{\cos x}{x^4-1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{x^4-1}\,dx$

Let $f(z) = \dfrac{e^{iz}}{z^4-1} = \dfrac{e^{iz}}{(z^2-1)(z^2+1)}$



$\oint_C f(z)\,dz = \left(\int_{C_R} + \int_{C_{\varepsilon1}} + \int_{C_{\varepsilon2}} + \int_{\Gamma}\right) f(z)\,dz$.

$C = C_R + C_{\varepsilon1} + C_{\varepsilon2} + \Gamma$ $\quad \int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$

$$\int_{\Gamma} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= 2\pi i \cdot \operatorname*{Res}_{z=i} f(z) + \pi i\left(\operatorname*{Res}_{z=-1} f(z) + \operatorname*{Res}_{z=1} f(z)\right)$$

$$\operatorname*{Res}_{z=i} f(z) = \left.\frac{e^{iz}}{(z^2-1)(z+i)}\right|_{z=i} = \frac{e^{-1}}{-2\cdot 2i} = \frac{i}{4e}$$

$$\operatorname*{Res}_{z=-1} f(z) = \left.\frac{e^{iz}}{(z-1)(z^2+1)}\right|_{z=-1} = \frac{e^{-i}}{-2\cdot 2} = -\frac{e^{-i}}{4}$$

$$\operatorname*{Res}_{z=1} f(z) = \left.\frac{e^{iz}}{(z+1)(z^2+1)}\right|_{z=1} = \frac{e^{i}}{2\cdot 2} = \frac{e^{i}}{4}$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4-1}\,dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x^4-1}\,dx = 2\pi i\cdot\frac{i}{4e} + \pi i\cdot\left(\frac{e^{i}}{4} - \frac{e^{-i}}{4}\right)$$

$$= \frac{-\pi}{2e} + \frac{\pi i}{4}\left[\cos 1 + i\sin 1 - (\cos 1 - i\sin 1)\right]$$

$$= \frac{-\pi}{2e} + \frac{\pi i}{4}\cdot 2i\sin 1 = \frac{-\pi}{2e} - \frac{\pi}{2}\sin 1 = -\frac{\pi}{2}\left(\frac{1}{e} + \sin 1\right)$$

$$\Rightarrow \int_0^{\infty} \frac{\cos x}{x^4-1}\,dx = -\frac{\pi}{4}\left(\frac{1}{e} + \sin 1\right)$$ #