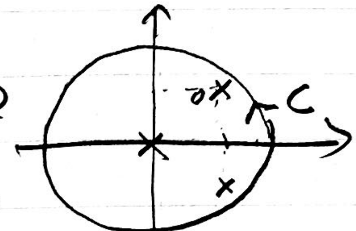
Complex Analysis Final 2019/1/11 DCChang

#1 (a) $f(z) = \dfrac{(z-3iz-2)^2}{z\cdot(z^2-2z+2)^5}$



$f(z)$ has zero $\frac{1+3i}{5}$ of order 2
and poles: 0 of order 1,
$1+i$ of order 5, $1-i$ of order 5 inside C.

$N_z = 2$, $N_p = 11$

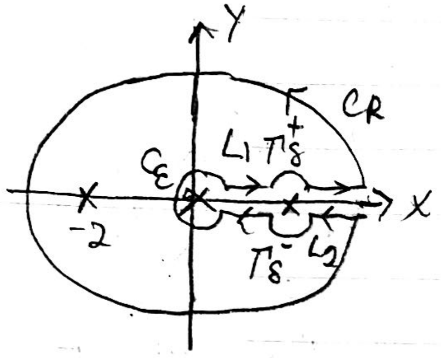
$$\oint_C \frac{f'(z)}{f(z)}\,dz = 2\pi i\cdot(N_z - N_p) = 2\pi i\cdot(-9) = \underline{-18\pi i} \ \#$$

(b) $f(z) = z^6 - 2iz^4 + (5-i)z^2 + 10$

$\because$ C encloses all of zeros of $f(z)$,
and $f(z)$ has 6 zeros (6 solutions)
in the complex plane,

$$\oint_C \frac{f'(z)}{f(z)}\,dz = 2\pi i\cdot 6 = \underline{12\pi i} \ \#$$

#2. $f(z) = \dfrac{z^{-1/2}}{z^2-4}$



$C = C_\varepsilon + C_R + L_1 + L_2 + \Gamma_\delta^+ + \Gamma_\delta^-$

$R \to \infty,\ \varepsilon \to 0,\ \delta \to 0$

$f(z)$ has poles at $-2, 0, 2$

$\oint_C f(z)\,dz = 2\pi i \cdot \operatorname*{Res}_{z=-2} f(z)$

① C_R : $R \to \infty$, $z = Re^{i\theta}$, $dz = Rie^{i\theta}d\theta$, $0 \le \theta \le 2\pi$

$$\left|\int_{C_R} f(z)\,dz\right| \le \int_{C_R} |f(z)|\,|dz|$$
$$= \lim_{R\to\infty} \int_0^{2\pi} \frac{R^{-\frac{1}{2}} \cdot |e^{-i\theta/2}|}{|R^2 e^{i2\theta} - 4|} R\,d\theta$$
$$= \lim_{R\to\infty} \int_0^{2\pi} \frac{R^{-\frac{1}{2}}}{R^2} \cdot R\,d\theta$$
$$= \lim_{R\to\infty} 2\pi \cdot \frac{1}{R^{3/2}} = 0$$

② C_ε : $\varepsilon \to 0$, $z = \varepsilon e^{i\theta}$, $dz = \varepsilon i e^{i\theta} d\theta$

$$\left|\int_{C_\varepsilon} f(z)\,dz\right| \le \int_{C_\varepsilon} |f(z)|\,|dz|$$
$$= \lim_{\varepsilon\to 0} \int_0^{2\pi} \frac{\varepsilon^{-\frac{1}{2}} |e^{-i\theta/2}|}{|\varepsilon^2 e^{i2\theta} - 4|} \varepsilon\,d\theta$$
$$= \lim_{\varepsilon\to 0} 2\pi \cdot \frac{\varepsilon^{\frac{1}{2}}}{4} = 0$$

③ L_1 : $z = x$, $dz = dx$

$$\int_{L_1} f(z)\,dz = \text{P.V.}\int_0^\infty \frac{x^{-\frac{1}{2}}}{x^2-4}\,dx$$

④ L_2 : $z = xe^{i2\pi}$, $dz = dx$

$$\int_{L_2} f(z)\,dz = \text{P.V.}\int_\infty^0 \frac{x^{-\frac{1}{2}} \cdot e^{-i\pi}}{x^2-4}\,dx = \underbrace{-e^{-i\pi}}_{=1} \cdot \left(\text{P.V.}\int_0^\infty \frac{x^{-\frac{1}{2}}}{x^2-4}\,dx\right)$$

⑤ $\int_{\Gamma_\delta^+} f(z)dz = -\pi i \cdot \operatorname{Res}_{z=2} f(z)$

$$= -\pi i \cdot \lim_{z \to 2} \left(\frac{z^{-\frac{1}{2}}}{z+2} \right) = -\frac{\pi i}{4} \cdot \frac{1}{\sqrt{2}}$$

⑥ $\int_{\Gamma_\delta^-} f(z)dz = -\pi i \cdot \operatorname{Res}_{z=2} f(z e^{i2\pi})$

$$= -\pi i \lim_{z \to 2} \left(\frac{z^{-\frac{1}{2}} \cdot e^{-i\pi}}{z+2} \right) = \frac{\pi i}{4} \cdot \frac{1}{\sqrt{2}}$$

⑦ $2\pi i \cdot \operatorname{Res}_{z=-2} f(z) = 2\pi i \lim_{z=-2} \frac{z^{-\frac{1}{2}}}{z-2}$

$$= 2\pi i \cdot \frac{(-2)^{-\frac{1}{2}}}{-4}$$

$$= -\frac{\pi i}{2} \cdot e^{-\frac{1}{2}\ln(-2)}$$

$$= -\frac{\pi i}{2} \cdot e^{(-\frac{1}{2}\ln 2 - \frac{1}{2} i\pi)}$$

$$= -\frac{\pi i}{2} \left(\frac{1}{\sqrt{2}} \cdot \left(\cos\frac{\pi}{2} - i \sin\frac{\pi}{2}\right)\right)$$

$$= -\frac{\pi i}{2} \cdot \frac{-i}{\sqrt{2}} = \frac{-\pi}{2\sqrt{2}}$$

$$\Rightarrow 2 \cdot \left(P.V. \int_0^\infty \frac{x^{-\frac{1}{2}}}{x^2-4} dx \right) = \frac{-\pi}{2\sqrt{2}}$$

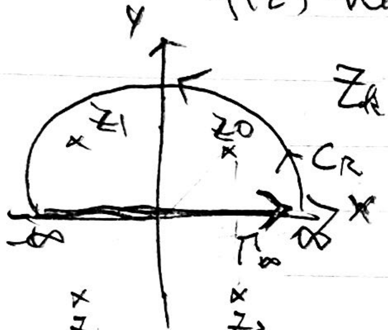
$$P.V. \int_0^\infty \frac{x^{-\frac{1}{2}}}{x^2-4} dx = \frac{-\pi}{4\sqrt{2}}$$

#

#3. (a)

Let $f(z) = \dfrac{z e^{i3z}}{z^4+4}$

$f(z)$ has four simple poles at



$z_k = \sqrt{2}\, e^{i(2k+1)\pi/4}$, $k = 0, 1, 2, 3$

$C = C_R + \Gamma_\infty$

$\oint_C f(z)\,dz = \overbrace{\int_{C_R} f(z)\,dz}^{\to 0} + \int_{\Gamma_\infty} f(z)\,dz$

$\Rightarrow 2\pi i \left[\operatorname*{Res}_{z=z_0} f(z) + \operatorname*{Res}_{z=z_1} f(z) \right] = \int_{\infty}^{\infty} \dfrac{x e^{i3x}}{x^4+4}\,dx$

$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \dfrac{z e^{i3z}}{4z^3}$ $\qquad z_0 = \sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$

$= \dfrac{e^{-3+3i}}{4z_0^2}$ $\qquad = 1+i$

$= \dfrac{e^{-3+3i}}{8i}$

$z_1 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right)$

$\operatorname*{Res}_{z=z_1} f(z) = \lim_{z \to z_1} \dfrac{e^{i3z}}{4z^2}$ $\qquad = -1+i$

$= \dfrac{e^{-3-3i}}{-8i}$

$\Rightarrow \int_{-\infty}^{\infty} \dfrac{x\cos 3x}{x^4+4}\,dx + i \int_{\infty}^{\infty} \dfrac{x \sin 3x}{x^4+4}\,dx$

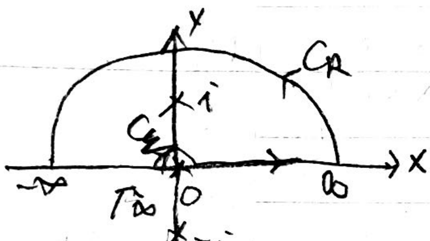
$= \dfrac{1}{8i}\left(e^{-3+3i} - e^{-3-3i}\right) \cdot 2\pi i$

$= \left(\dfrac{1}{8i} \cdot e^{-3} \cdot 2i \sin 3\right) \cdot 2\pi i = \dfrac{i\pi \sin 3}{2e^3}$

$\Rightarrow \int_{-\infty}^{\infty} \dfrac{x^3 \sin 3x}{x^4+4}\,dx = \underline{\dfrac{\pi \sin 3}{2e^3}}$ #

(b) $\int_0^\infty \frac{\sin(2x)}{x(x^2+1)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin(2x)}{x(x^2+1)^2}\,dx$

Let $f(z) = \frac{e^{i2z}}{z(z^2+1)^2}$



$f(z)$ has a simple pole at 0, two poles of order 2 at $\pm i$

$C = C_R + T_\infty + C_\varepsilon$

$$\oint_C f(z)\,dz = \int_{C_R + C_\varepsilon + T_\infty} f(z)\,dz$$

① By Jordan's Lemma,

$$\int_{C_R} f(z)\,dz = 0$$

② $f(z)$ has a simple pole at $z=0$.

$$\int_{C_\varepsilon} f(z)\,dz = -\pi i \operatorname*{Res}_{z=0} f(z) = -\pi i \left.\frac{e^{i2z}}{(z^2+1)^2}\right|_{z=0} = -\pi i$$

③ $$\oint_C f(z)\,dz = 2\pi i \cdot \operatorname*{Res}_{z=i} f(z) = 2\pi i \cdot \frac{d}{dz}\left.\frac{e^{i2z}}{z(z+i)^2}\right|_{z=i}$$

$$= 2\pi i \cdot \left.\frac{2ie^{i2z}z(z+i)^2 - e^{i2z}\left[(z+i)^2 + 2z(z+i)\right]}{z^2(z+i)^4}\right|_{z=i}$$

$$= 2\pi i \cdot \frac{2i\cdot e^{-2}\cdot i\cdot(-4) - e^{-2}\cdot(-4-4)}{-16} = -\frac{2\pi i}{e^2}$$

④ $$\int_{T_\infty} f(z)\,dz = p.v.\int_{-\infty}^{\infty} \frac{\sin 2x}{x(x^2+1)^2}\,dx$$

$$\Rightarrow p.v.\left(\int_{-\infty}^{\infty} \frac{\cos 2x}{x(x^2+1)^2}\,dx + i\int_{-\infty}^{\infty} \frac{\sin 2x}{x(x^2+1)^2}\,dx\right) = i\pi\left(1 - \frac{2}{e^2}\right)$$

$$\Rightarrow p.v.\int_0^\infty \frac{\sin 2x}{x(x^2+1)^2}\,dx = \underline{\pi\left(\frac{1}{2} - \frac{1}{e^2}\right)}$$ ##

(c) Let $z = e^{i\theta}$, $0 \le \theta \le 2\pi$ $\quad dz = iz\,d\theta$

$$\cos 2\theta = \frac{z^2+z^{-2}}{2}, \quad \sin\theta = \frac{z-z^{-1}}{2i}$$

$$f(z) = \frac{\cos 2\theta}{5-4\sin\theta} = \frac{\frac{z^2+z^{-2}}{2}}{5-4\cdot\frac{z-z^{-1}}{2i}}\cdot\frac{1}{iz}$$

$$= \frac{z^4+1}{2iz^2(2iz^2+5z-2i)}$$

$f(z)$ has a 2nd-order pole at $z=0$ and two simple poles at $z=\frac{i}{2}$ and $2i$.

$$\int_0^{2\pi}\frac{\cos 2\theta}{5-4\sin\theta}d\theta = \oint_{|z|=1} f(z)dz$$

$$= 2\pi i\cdot\left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=i/2} f(z)\right)$$

① $$\operatorname*{Res}_{z=0} f(z) = \frac{1}{2i}\cdot\lim_{z\to 0}\frac{d}{dz}\left(\frac{z^4+1}{2iz^2+5z-2i}\right)$$

$$= \frac{1}{2i}\cdot\left.\frac{4z^3(2iz^2+5z-2i)-(z^4+1)\cdot(4iz+5)}{(2iz^2+5z-2i)^2}\right|_{z=0}$$

$$= \frac{1}{2i}\cdot\frac{5}{4} = -\frac{5i}{8}$$

② $$\operatorname*{Res}_{z=i/2} f(z) = \left(\frac{z^4+1}{2iz^2}\right)_{z=\frac{i}{2}}\cdot\left.\frac{1}{(2iz^2+5z-2i)'}\right|_{z=\frac{i}{2}}$$

$$= \frac{\frac{1}{16}+1}{-i/2}\cdot\frac{1}{3} = \frac{17}{24}i$$

$$\int_0^{2\pi}\frac{\cos 2\theta}{5-4\sin\theta}d\theta = 2\pi i\cdot\left(-\frac{5}{8}i+\frac{17}{24}i\right)$$

$$= -\frac{\pi}{6}$$

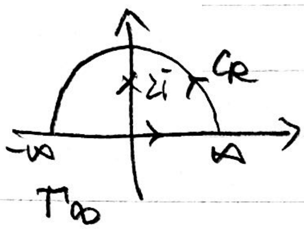
#

(d)

$$\text{p.v.}\int_{-\infty}^{\infty}\frac{\sin 3x}{x-2i}\,dx = \text{p.v.}\int_{-\infty}^{\infty}\frac{\frac{1}{2i}(e^{i3x}-e^{-i3x})}{x-2i}\,dx$$

$$=\frac{1}{2i}\left(\text{p.v.}\int_{-\infty}^{\infty}\frac{e^{i3x}}{x-2i}\,dx - \text{p.v.}\int_{-\infty}^{\infty}\frac{e^{-i3x}}{x-2i}\,dx\right)$$

① Let $f(z)=\dfrac{e^{i3z}}{z-2i}$



$C = C_R + \Gamma_\infty$ $\quad \oint_C f(z)dz = \int_{C_R+\Gamma_\infty} f(z)dz$

By Jordan's Lemma

$$\int_{C_R} f(z)dz = 0.$$

$$\Rightarrow \int_{\Gamma_\infty} f(z)dz = \text{p.v.}\int_{-\infty}^{\infty}\frac{e^{i3x}}{x-2i}\,dx$$

$$=\oint_C f(z)\,dz$$

$$=2\pi i\cdot \operatorname*{Res}_{z=2i} f(z)$$

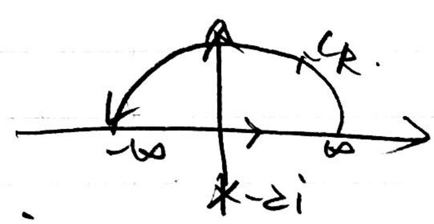
$$=2\pi i\cdot e^{i3z}\Big|_{z=2i} = \frac{2\pi i}{e^6}$$

② $\text{p.v.}\int_{-\infty}^{\infty}\dfrac{e^{-i3x}}{x-2i}\,dx$

$$=\text{p.v.}\int_{\infty}^{-\infty}\frac{e^{i3y}}{-(y+2i)}(-dy)\,,\quad y=-x$$

$$=-\,\text{p.v.}\int_{-\infty}^{\infty}\frac{e^{i3x}}{x+2i}\,dx$$

$$=0.$$



$$\therefore\ \text{p.v.}\int_{-\infty}^{\infty}\frac{\sin 3x}{x-2i}\,dx = \frac{1}{2i}\cdot\frac{2\pi i}{e^6}$$

$$=\frac{\pi}{e^6}\ \#$$