Complex Analysis 2018 Midterm DCChang 2018/12/7

1. (a)

$$r^2 u_{rr} = r^2 \frac{\partial}{\partial r}\left(\frac{1}{r} v_\theta\right) = r^2 \cdot \left(-\frac{1}{r^2} v_\theta + \frac{1}{r} v_{\theta r}\right)$$
$$= -v_\theta + r \cdot v_{\theta r}$$

$$r u_r = v_\theta$$

$$u_{\theta\theta} = \frac{\partial}{\partial \theta}(-r v_r) = -r v_{r\theta} = -r v_{\theta r}$$

$$\Rightarrow r^2 u_{rr} + r u_r + u_{\theta\theta} = -v_\theta + r v_{\theta r} + v_\theta - r v_{\theta r}$$
$$= 0 \quad \#$$

(b)

$$u = r^3 \cos 3\theta$$

$$v_\theta = r u_r = r \cdot 3r^2 \cos 3\theta = 3r^3 \cos 3\theta$$

$$v = r^3 \sin 3\theta + f(r)$$

$$v_r = 3r^2 \sin 3\theta + f'(r)$$
$$= -\frac{1}{r} u_\theta$$
$$= -\frac{1}{r}(-3r^3 \sin 3\theta)$$
$$= 3r^2 \sin 3\theta$$

$$\therefore f'(r) = 0, \quad f(r) = c$$

$$\Rightarrow v(r, \theta) = r^3 \sin 3\theta + c \quad \#$$

2.

Let $w = \sinh^{-1} z$

$$z = \sinh w = \frac{e^{w} - e^{-w}}{2}$$

$$e^{2w} - 2z e^{w} - 1 = 0$$

$$e^{w} = z + (z^2+1)^{\frac{1}{2}}$$

$$\Rightarrow w = \sinh^{-1} z = \log\left[z + (z^2+1)^{\frac{1}{2}}\right] \#$$

Similarly,

Let $w = \tanh^{-1} z$

$$z = \tanh w = \frac{\sinh w}{\cosh w} = \frac{e^{w} - e^{-w}}{e^{w} + e^{-w}}$$

$$= \frac{e^{2w} - 1}{e^{2w} + 1}$$

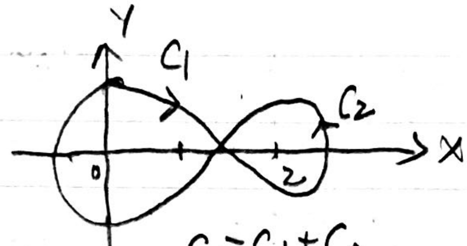
$$z e^{2w} + z = e^{2w} - 1$$

$$(z-1) e^{2w} = -(z+1)$$

$$e^{2w} = -\frac{z+1}{z-1}, \quad z \neq \pm 1$$

$$w = \tanh^{-1} z = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right) \#$$

3.



$C=C_1+C_2$

$$\oint_C \frac{3z+1}{z(z-2)^3}\,dz$$

$$=\oint_{C_1} \frac{3z+1/(z-2)^3}{z}\,dz$$

$$+\oint_{C_2} \frac{3z+1/z}{(z-2)^3}\,dz$$

$$=-2\pi i\cdot \frac{3z+1}{(z-2)^3}\Big|_{z=0}+\frac{2\pi i}{2!}\cdot\left(\frac{3z+1}{z}\right)''\Big|_{z=2}$$

$$=-2\pi i\cdot\frac{1}{-8}+\pi i\cdot\frac{2}{z^3}\Big|_{z=2}=\frac{\pi i}{4}+\frac{\pi i}{4}=\frac{\pi i}{2}\ \#$$

4.

(a)

$$a_n=\frac{1}{n^2(3+4i)^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{n^2(3+4i)^n}{(n+1)^2(3+4i)^{n+1}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2\cdot(3+4i)}\right|=\frac{1}{5}$$

radius $R=5$ #

(b)

$$a_n=\frac{1}{n^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{n^n}{(n+1)^{n+1}}\right|$$

$$=\lim_{n\to\infty}\frac{n^n}{(n+1)^n\cdot(n+1)}=0$$

radius $=\infty$ # (converges absolutely for all z.)

(c) $a_n = \dfrac{n!}{(2n)^n}$ for z^3

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\dfrac{(n+1)!}{2^{n+1}\cdot(n+1)^{n+1}}}{\dfrac{n!}{(2n)^n}}$$

$$= \lim_{n\to\infty}\frac{2^n\cdot n^n\cdot(n+1)!}{2^{n+1}\cdot(n+1)^{n+1}\cdot n!}$$

$$= \lim_{n\to\infty}\frac{1}{2}\cdot\left(\frac{n}{n+1}\right)^n$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n = \frac{1}{2e}$$

$\therefore$ radius of z^3 is $2e$

$\Rightarrow$ radius of z is $\sqrt[3]{2e}$ #

5. Let $z_n = \left(\dfrac{z-1}{z+2}\right)^n$

By Ratio Test,

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left|\frac{(z-1)^{n+1}}{(z+2)^{n+1}}\cdot\frac{(z+2)^n}{(z-1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{z-1}{z+2}\right|$$

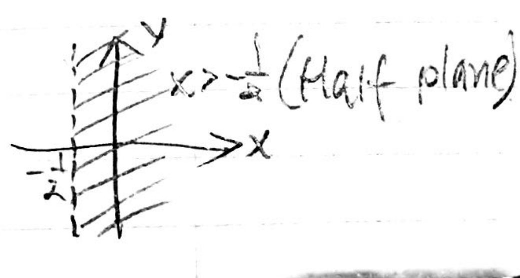
If z_n converges, $\left|\dfrac{z-1}{z+2}\right| < 1$

$\Rightarrow |z-1| < |z+2|$, $z = x+iy$

$\Rightarrow (x-1)^2 + y^2 < (x+2)^2 + y^2$

$\Rightarrow -6x < 3$

$\Rightarrow x > -\frac{1}{2}$ ##



6. (a)

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots\cdots$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\cdots$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\cdots$$

$$\cot z = \frac{\cos z}{\sin z} = \frac{1 - \frac{z^2}{2} + \frac{z^3}{24} - \cdots\cdots}{z\left(1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots\cdots\right)}$$

$$= \frac{1}{z}\left(1 - \frac{1}{3}z^2 - \frac{1}{45}z^4 - \cdots\right)$$

(b) $$\frac{\cot z}{z^2} = \frac{1}{z^3}\left(1 - \frac{1}{3}z^2 - \frac{1}{45}z^4 - \cdots\cdots\right)$$

$$\oint_C \frac{\cot z}{z^2}\,dz = \oint_C \left(\frac{1}{z^3} - \frac{1}{3z} - \frac{z}{45} - \cdots\right) dz$$

$$= -\frac{1}{3}\oint_C \frac{1}{z}\,dz = -\frac{1}{3} \times 2\pi i = -\frac{2\pi i}{3}$$ #

7. $f(z) = \dfrac{z+1}{(z-1)(z-4)^3}$

$$= \frac{1}{(z-4)^3} \cdot \frac{z+1}{z-1} = \frac{1}{(z-4)^3} \cdot \left(1 + \frac{2}{z-1}\right)$$

For $|z-4| > 3$,

$$\frac{2}{z-1} = \frac{2}{(z-4)+3} = \frac{2}{z-4} \cdot \frac{1}{1+\frac{3}{z-4}}$$

$$= \frac{2}{z-4} \cdot \sum_{n=0}^{\infty} \left(-\frac{3}{z-4}\right)^n \quad \left|\frac{3}{z-4}\right| < 1$$

$$f(z) = \frac{1}{(z-4)^3} \left[1 + \frac{2}{z-4} \cdot \sum_{n=0}^{\infty} \left(-\frac{3}{z-4}\right)^n\right]$$

$$= \frac{1}{(z-4)^3} + \frac{2}{(z-4)^4} - \frac{6}{(z-4)^5} + \frac{18}{(z-4)^6} - \cdots$$

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8. $f(z) = \dfrac{z}{(z-1)(z-3)} = \dfrac{1}{z-1} \cdot \dfrac{z}{z-3} = \dfrac{1}{z-1}\left(1 + \dfrac{3}{z-3}\right)$

For $0 < |z-1| < 2$,

$$\frac{3}{z-3} = \frac{3}{(z-1)-2} = -\frac{3}{2} \cdot \frac{1}{1-\frac{z-1}{2}}, \quad \left|\frac{z-1}{2}\right| < 1$$

$$= -\frac{3}{2} \sum_{n=0}^{\infty} \left(\frac{z-1}{2}\right)^n$$

$$f(z) = \frac{1}{z-1} \cdot \left[1 - \frac{3}{2} \sum_{n=0}^{\infty} \left(\frac{z-1}{2}\right)^n\right]$$

$$= \frac{1}{z-1} \left[1 - \frac{3}{2} - \frac{3}{2}\left(\frac{z-1}{2}\right) - \frac{3}{2}\left(\frac{z-1}{2}\right)^2 - \frac{3}{2}\left(\frac{z-1}{2}\right)^3 \cdots\right]$$

$$= \frac{-1/2}{z-1} - \frac{3}{4} - \frac{3}{8}(z-1) - \frac{3}{16}(z-1)^2 - \cdots$$

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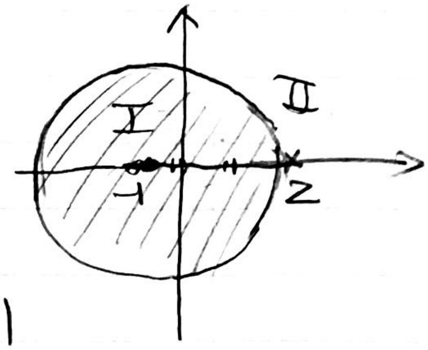
9. $f(z)=\dfrac{3}{2+z-z^2}=\dfrac{1}{1+z}+\dfrac{1}{2-z}$

(I) $|z+1|<3$



$$\frac{1}{2-z}=\frac{1}{3-(z+1)}$$

$$=\frac{1/3}{1-\frac{z+1}{3}} \qquad \left|\frac{z+1}{3}\right|<1$$

$$=\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

$$f(z)=\frac{1}{z+1}+\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

$$=\frac{1}{z+1}+\frac{1}{3}+\frac{1}{9}(z+1)+\frac{1}{27}(z+1)^2+\cdots \quad \#$$

(II) $|z+1|>3$

$$\frac{1}{2-z}=\frac{1}{3-(z+1)}=\frac{-1}{z+1}\cdot\frac{1}{1-\frac{3}{z+1}}$$

$$=\frac{-1}{z+1}\cdot\sum_{n=0}^{\infty}\left(\frac{3}{z+1}\right)^n, \quad \left|\frac{3}{z+1}\right|<1$$

$$f(z)=\frac{1}{z+1}-\frac{1}{z+1}\cdot\sum_{n=0}^{\infty}\left(\frac{3}{z+1}\right)^n$$

$$=\frac{1}{z+1}\left[1-\sum_{n=0}^{\infty}\left(\frac{3}{z+1}\right)^n\right]$$

$$=\frac{1}{z+1}\left(-\frac{3}{z+1}-\frac{9}{(z+1)^2}-\frac{27}{(z+1)^3}-\frac{81}{(z+1)^4}\cdots\right)$$

$$=-\frac{3}{(z+1)^2}-\frac{9}{(z+1)^3}-\frac{27}{(z+1)^4}-\frac{81}{(z+1)^5}\cdots \quad \#$$