Complex Anlysis Quiz-2 Solution

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1. (10%)

$$0+0-\frac{1}{3}(-2\pi i)+0=\frac{2\pi i}{3}$$

2. (15%)

Let $C = C_1 + C_2 + C_3 + C_4$, where

$$\begin{array}{ll} C_1 : z(t) = t & 0 \le t \le 1 \\ C_2 : z(t) = 1 + it & 0 \le t \le 1 \\ C_3 : z(t) = 1 - t + i & 0 \le t \le 1 \\ C_4 : z(t) = i(1-t) & 0 \le t \le 1 \end{array}$$

$$\begin{aligned}\int_C \bar{z}^2 dz = \int_0^1 t^2 dt + \int_0^1 (1-it)^2 (i) dt \\ + \int_0^1 (1-t-i)^2(-1)dt + \int_0^1 (-i)^2(1-t)^2(-i)dt \\ = 2(1+i)\end{aligned}$$

3. (15%)

$$\left|\int_\Gamma \mathrm{Log} z dz\right| \le \max_{z \text{ on } \Gamma}\left|\mathrm{Log}|z| + i\mathrm{Arg}\, z\right| l(\Gamma) \le (|\mathrm{Log}\ e| + |i\pi|) \times \frac{\pi \mathrm{e}}{2} = \frac{\pi e + \pi^2 e}{2}$$

4. (15%)

$$I = \int \frac{1}{z^2+1} dz = \frac{i}{2}\int \frac{1}{z+i} dz - \frac{i}{2}\int \frac{1}{z-i} dz$$

Along Γ_1: $I = \frac{i}{2}(0) - \frac{i}{2}(2\pi i) = \pi$

Along Γ_2: $I = \frac{i}{2}(2\pi i) - \frac{i}{2}(2\pi i) = 0$

Along Γ_3: $I = \frac{i}{2}(2\pi i) - \frac{i}{2}(0) = -\pi$

5. (15%)

$$\begin{aligned}\oint_{|z-i|=r} \frac{e^{iz}/(z+i)^2}{(z-i)^2} dz + \oint_{|z+i|=r} \frac{e^{iz}/(z-i)^2}{(z+i)^2} dz \text{ where } r < 2 \\ = 2\pi i\left(-\frac{e^{-1}i}{2}\right) + 2\pi i(0) = \frac{\pi}{e}\end{aligned}$$

6. (10%)

$$I(R) = \oint_{|z|=R} \frac{z\,dz}{(z-1)^3}$$

$$\because \left|\frac{z}{(z-1)^3}\right| = \frac{|z|}{|z-1|^3} \le \frac{|z|}{||z|-1|^3} = \frac{R}{|R-1|^3}$$

By ML-inequality,

$$\left|\oint_{|z|=R} \frac{z\,dz}{(z-1)^3}\right| \le \frac{R}{|R-1|^3} \cdot 2\pi R$$

(a) $\lim\limits_{R\to\infty} 2\pi R \cdot \frac{R}{(R-1)^3} = \lim\limits_{R\to\infty} \frac{4\pi}{6R} = 0$

(b) $\lim\limits_{R\to 0} 2\pi R \frac{R}{(1-R)^3} = \lim\limits_{R\to 0} 2\pi R^2 = 0$

7. (10%)

$$f(z) = \frac{\cos z}{z-3}$$

$$f''(z) = \frac{2(z-3)\sin z - (z^2 - 6z + 7)\cos z}{(z-3)^3}$$

$$\frac{2\pi i}{2} \times f''(0) = \pi i \times \frac{7}{27} = \frac{7\pi i}{27}$$

8. (10%)

With $P(z) = c(z - z_1)(z - z_2)\cdots(z - z_n)$, apply the product rule to get

$$\frac{P'(z)}{P(z)} = \frac{\sum_{k=1}^{n} c(z-z_1)\cdots(z-z_{k-1})(z-z_{k+1})\cdots(z-z_n)}{c(z-z_1)(z-z_2)\cdots(z-z_n)}$$

$$= \sum_{k=1}^{n} \frac{1}{z - z_k}$$

Now $\frac{1}{2\pi i}\int_\Gamma \frac{dz}{z - z_k} = \begin{cases} 1 & \text{if } z_k \text{ is inside } \Gamma \\ 0 & \text{otherwise} \end{cases}$

Thus

$$\frac{1}{2\pi i}\int_\Gamma \frac{P'(z)}{P(z)}dz = \frac{1}{2\pi i}\sum_{k=1}^{n}\int_\Gamma \frac{dz}{z - z_k} = n$$