Complex Analysis Quiz-3, 2018 DCChang.

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

1.

a) $f(z) = \dfrac{1}{(\sqrt{2}\cos z - 1 + z - \frac{\pi}{4})^2}$

$$\begin{aligned}
h(z) &= \sqrt{2}\cos z - 1 + z - \tfrac{\pi}{4} \\
&= \sqrt{2}\cos\left[(z-\tfrac{\pi}{4}) + \tfrac{\pi}{4}\right] - 1 + (z-\tfrac{\pi}{4}) \\
&= \sqrt{2}\cos\tfrac{\pi}{4}\cos(z-\tfrac{\pi}{4}) - \sqrt{2}\sin\tfrac{\pi}{4}\sin(z-\tfrac{\pi}{4}) - 1 + (z-\tfrac{\pi}{4}) \\
&= \cos(z-\tfrac{\pi}{4}) - \sin(z-\tfrac{\pi}{4}) + (z-\tfrac{\pi}{4}) - 1 \\
&= \cancel{1} - (z-\tfrac{\pi}{4})^2/2 + (z-\tfrac{\pi}{4})^4/4! - \cdots \\
&\quad - \left[\cancel{(z-\tfrac{\pi}{4})} - (z-\tfrac{\pi}{4})^3/6 + \cdots\right] + \cancel{(z-\tfrac{\pi}{4})} - \cancel{1} \\
&= -\frac{(z-\frac{\pi}{4})^2}{2} + \frac{(z-\frac{\pi}{4})^3}{6} + \cdots \\
&= -\frac{1}{2}(z-\tfrac{\pi}{4})^2\left[1 - \frac{(z-\frac{\pi}{4})}{3} + \cdots\right]
\end{aligned}$$

$h(z)$ has a 2nd order of zero at $z=\frac{\pi}{4}$

$\Rightarrow f(z)$ has a <u>4th order of</u> pole at $z=\frac{\pi}{4}$.

b). Let $f(z) = \dfrac{g(z)}{(z-z_0)^4}$, $z_0=\frac{\pi}{4}$, $g(z)$ is analytic at z_0

$$\begin{aligned}
f'(z) &= \frac{g'(z)(z-z_0)^4 - g(z)\cdot 4(z-z_0)^3}{(z-z_0)^8} \\
&= \frac{g'(z)}{(z-z_0)^4} - 4\cdot\frac{g(z)}{(z-z_0)^5}
\end{aligned}$$

$\dfrac{f'(z)}{f(z)} = \dfrac{g'(z)}{g(z)} - \dfrac{4}{z-z_0}$, z_0 inside $C: |z-\frac{\pi}{4}|=1$

$\oint_C \dfrac{f'(z)}{f(z)}dz = -4\oint_C \dfrac{dz}{z-z_0} = -4\times 2\pi i = \underline{-8\pi i}$ #

2.

(a) $\cot z = \frac{\cos z}{\sin z} = \frac{1 - z^2/2! + z^4/4! - \cdots}{z - z^3/3! + z^5/5! - \cdots}$

$= \frac{1}{z}\left(1 - \frac{1}{3}z^2 + \cdots\right)$

$\oint_{|z|=1} \cot z \, dz = \oint_{|z|=1} \frac{1}{z} dz = \underline{2\pi i}$ #

(b) $\csc z = \frac{1}{\sin z} = \frac{1}{z - \frac{z^3}{6} + \cdots}$

$= \frac{1}{z}\left(1 + \frac{z^2}{6} + \frac{7z^4}{360} + \cdots\right)$

$\frac{\csc z}{z^2} = \frac{1}{z^3} + \frac{1}{6z} + \frac{7z}{360} + \cdots$

$\oint_{|z|=1} \frac{\csc z}{z^2} dz = \oint_{|z|=1} \frac{1}{6z} dz = \frac{1}{6} \times 2\pi i = \underline{\frac{\pi i}{3}}$ #

(c) Within $|z|=4$, $\sin z$ has 3 simple poles at $z = -\pi, 0, \pi$.

$\operatorname*{Res}_{z=-\pi} \frac{z-1}{\sin z} + \operatorname*{Res}_{z=0} \frac{z-1}{\sin z} + \operatorname*{Res}_{z=\pi} \frac{z-1}{\sin z}$

$= \frac{z-1}{\cos z}\Big|_{z=-\pi} + \frac{z-1}{\cos z}\Big|_{z=0} + \frac{z-1}{\cos z}\Big|_{z=\pi}$

$= -(\cancel{-\pi}-1) + (-1) - (\cancel{\pi}-1) = 1$

$\oint_{|z|=4} \frac{z-1}{\sin z} dz = 2\pi i \cdot 1 = \underline{2\pi i}$ #

(d) $\left(\frac{z-1}{z+1}\right)^3$ has a 3rd order of pole at $z=-1$

$\operatorname*{Res}_{z=-1} \left(\frac{z-1}{z+1}\right)^3 = \frac{1}{2!} \frac{d^2}{dz^2}\left[(z+1)^3 \cdot \frac{(z-1)^3}{(z+1)^3}\right] = \frac{1}{2} \frac{d}{dz^2}(z-1)^3 = -6$

$\oint_{|z|=2} \left(\frac{z-1}{z+1}\right)^3 dz = 2\pi i \cdot (-6) = \underline{-12\pi i}$ #

3.

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$P.V. \int_{-\infty}^{\infty} \frac{\cos x}{x-w}\,dx = P.V. \int_{-\infty}^{\infty} \frac{e^{ix}}{2(x-w)}\,dx + P.V. \int_{-\infty}^{\infty} \frac{e^{-ix}}{2(x-w)}\,dx$$

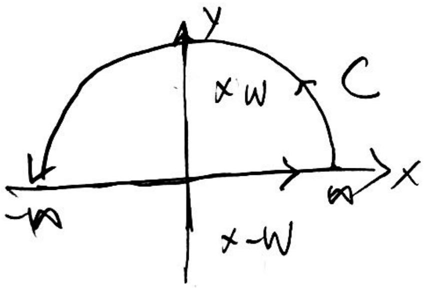
$$P.V. \int_{-\infty}^{\infty} \frac{e^{-ix}}{2(x-w)}\,dx = P.V. \int_{\infty}^{-\infty} \frac{e^{ix'}}{2(-x'-w)}(-dx') \quad , \quad x = -x'$$

$$= P.V. \int_{-\infty}^{\infty} \frac{-e^{ix}}{2(x+w)}\,dx$$

$$P.V. \int_{-\infty}^{\infty} \frac{\cos x}{x-w}\,dx = \oint_C \frac{e^{iz}}{2(z-w)}\,dz - \oint_C \frac{e^{iz}}{2(z+w)}\,dz$$

C includes the x-axis from $-\infty$ to ∞.

① $\mathrm{Im}\, w > 0$ (upper-half plane).

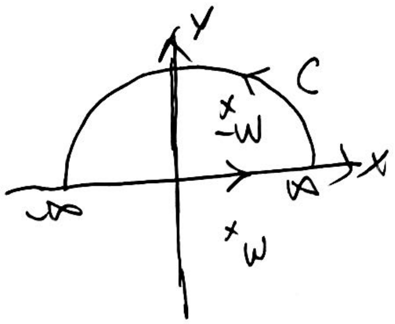


$$\oint_C \frac{e^{iz}}{2(z-w)}\,dz - \oint_C \frac{e^{iz}}{2(z+w)}\,dz$$

$$= \oint_C \frac{e^{iz}}{2(z-w)}\,dz$$

$$= 2\pi i \cdot \underset{z=w}{\mathrm{Res}} \frac{e^{iz}}{2(z-w)} = \underline{\pi i \cdot e^{iw}} \quad \#$$

② $\mathrm{Im}\, w < 0$ (lower-half plane)



$$\oint_C \frac{e^{iz}}{2(z-w)}\,dz - \oint_C \frac{e^{iz}}{2(z+w)}\,dz$$

$$= -\oint_C \frac{e^{iz}}{2(z+w)}\,dz$$

$$= -2\pi i \cdot \underset{z=-w}{\mathrm{Res}} \frac{e^{iz}}{2(z+w)} = \underline{-\pi i \cdot e^{-iw}} \quad \#$$

4. (a)

$$\int_0^{\pi} \frac{d\theta}{(3+2\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(3+2\cos\theta)^2}$$

Let $z = e^{i\theta}$, $0 \le \theta \le 2\pi$

$\cos\theta = \dfrac{z+z^{-1}}{2}$, $dz = ie^{i\theta}d\theta = iz\,d\theta$

$$f(z) = \frac{1}{\left(3+2\cdot\frac{z+z^{-1}}{2}\right)^2}\cdot\frac{1}{iz}$$

$$= \frac{1}{iz}\cdot\frac{1}{(3+z+z^{-1})^2} = \frac{1}{i}\cdot\frac{z}{(z^2+3z+1)^2}$$

$$= \frac{1}{i}\,\frac{z}{(z-z_0)^2(z-z_1)^2}, \quad z_0 = \frac{-3+\sqrt{5}}{2}, \quad z_1 = \frac{-3-\sqrt{5}}{2}$$

where z_0 is inside $|z|=1$.

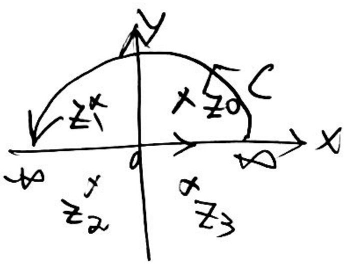
$$\int_0^{\pi} \frac{d\theta}{(3+2\cos\theta)^2} = \frac{1}{2}\oint_{|z|=1} f(z)\,dz$$

$$= \frac{2\pi i}{2i}\operatorname*{Res}_{z=z_0} f(z) = \pi\cdot\lim_{z\to z_0}\frac{d}{dz}\cdot\frac{z}{(z-z_1)^2}$$

$$= \pi\cdot\frac{-(z_0+z_1)}{(z_0-z_1)^3}$$

$$= \pi\cdot\left(\frac{3}{5\sqrt{5}}\right) = \frac{3\pi\sqrt{5}}{25} \quad \#$$

4. (b) $\int_0^\infty \frac{x^2+1}{x^4+1}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2+1}{x^4+1}dx = \frac{1}{2}\oint_C \frac{z^2+1}{z^4+1}dz$



$f(z) = \frac{z^2+1}{z^4+1}$

$= \prod_{n=0}^{3}\frac{1}{(z-z_n)}\cdot(z^2+1)$

$z_n = (-1)^{\frac{1}{4}} = e^{i(\pi+2k\pi)4}$, $k=0,1,2,3$.

$z_0 = e^{i\pi/4}$, $z_1 = e^{i3\pi/4}$ are two simple poles in the upper half-plane

$\operatorname*{Res}_{z=z_0} f(z) + \operatorname*{Res}_{z=z_1} f(z)$

$= \left.\frac{z^2+1}{4z^3}\right|_{z=z_0} + \left.\frac{z^2+1}{4z^3}\right|_{z=z_1}$

$= \frac{e^{i\pi/2}+1}{4e^{i3\pi/4}} + \frac{e^{i3\pi/2}+1}{4e^{i\pi/4}}$

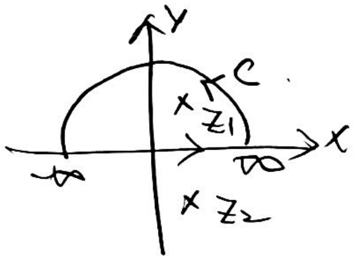
$= \frac{1+i}{4(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})} + \frac{1-i}{4(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})}$

$= \frac{1}{\sqrt{2}i}$

$\int_0^\infty \frac{x^2+1}{x^4+1}dx = \frac{1}{2}\cdot 2\pi i\cdot\frac{1}{\sqrt{2}i} = \frac{\pi}{\sqrt{2}}$ ※

4. (c)

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{x}{(x^2+4x+13)^2}\,dx = \oint_C \frac{z}{(z^2+4z+13)^2}\,dz$$



$$f(z) = \frac{z}{(z^2+4z+13)^2}$$

$$= \frac{z}{(z-z_1)^2(z-z_2)^2}$$

$$z_1 = -2+\sqrt{4-13} = -2+3i$$

$$z_2 = -2-3i$$

where z_1 is inside C.

$$\operatorname*{Res}_{z=z_1} f(z) = \lim_{z\to z_1} \frac{d}{dz}(z-z_1)^2 f(z)$$

$$= \lim_{z\to z_1} \frac{d}{dz} \frac{z}{(z-z_2)^2}$$

$$= -\frac{z_1+z_2}{(z_1-z_2)^3}$$

$$= -\frac{-4}{(6i)^3} = -\frac{4}{36\times 6i} = -\frac{1}{54i}$$

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{x}{(x^4+4x+13)^2}\,dx = 2\pi i \operatorname*{Res}_{z=z_1} f(z)$$

$$= -\frac{\pi}{27}$$

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