CO2013: Complex Analysis, Midterm, Fall 2018

Department of Communication Engineering, National Central University
Prof. Dah-Chung Chang (office: E1-311, e-mail: dcchang@ce.ncu.edu.tw)

Notice:
a) Term grading policy: Midterm×25%.
b) Total 100 points (2 pages, see the next page for problems 6-9.) in this exam.
c) Exam Time: 1:00PM−2:50PM, Dec. 7, 2018.

1. Suppose $f(z) = u(r,\theta) + iv(r,\theta)$ is analytic in a domain D not containing the origin. The Cauchy-Riemann equations are in the form $ru_r = v_\theta$ and $rv_r = -u_\theta$.
   (a) (8 pts) Show that $u(r,\theta)$ satisfies Laplace's equation in polar coordinates:
$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$
   (b) (8 pts) Let $u(r,\theta) = r^3\cos 3\theta$. Find a harmonic conjugate $v(r,\theta)$ of $u(r,\theta)$.

2. (10 pts) Show that $\sinh^{-1} z = \log\left[z+(z^2+1)^{1/2}\right]$ and $\tanh^{-1} z = \dfrac{1}{2}\log\left(\dfrac{1+z}{1-z}\right)$, $z \neq \pm 1$.

3. (8 pts) Evaluate the integral
$$\oint_C \frac{3z+1}{z(z-2)^3}dz,$$
   where C is the figure-eight contour depicted in Fig. 1.

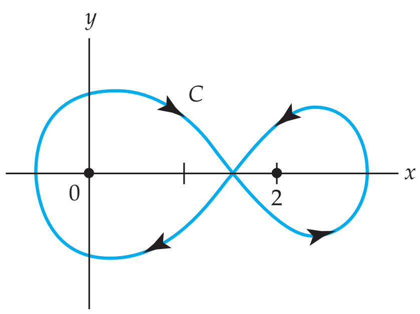

Fig. 1: Problem 3.

4. (15 pts) Find the radius of convergence in the following series:
   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{(z+3i)^n}{n^2(3+4i)^n}$, (b) $\displaystyle\sum_{n=1}^{\infty}\frac{z^n}{n^n}$, (c) $\displaystyle\sum_{n=0}^{\infty}\frac{n!}{(2n)^n}z^{3n}$.

5. (8 pts) Find the region in the complex plane for which $\displaystyle\sum_{n=0}^{\infty}\left(\frac{z-1}{z+2}\right)^n$ converges.

6. (a) (10 pts) Find the Taylor series of $\sin z$ and $\cos z$, and use them to write out the first three terms of the power series of $\cot z$ in powers of z.
   (b) (5 pts) Using the result obtained from (a), calculate

$$\oint_C \frac{\cot z}{z^2} dz,$$

   where $C : |z| = 1$.

> Notice: For problems 7-9, you need to explicitly write out at least the first four terms of the power series as your answer to each question. For example,
>
> $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots.$$

7. (8 pts) Find the Laurent series for

$$f(z) = \frac{z+1}{(z-1)(z-4)^3}$$

   with center 4 in the domain $|z-4| > 3$.

8. (8 pts) Find the Laurent series for

$$f(z) = \frac{z}{(z-1)(z-3)}$$

   with center 1 in the domain $0 < |z-1| < 2$.

9. (12 pts) Find the power series representation for

$$f(z) = \frac{3}{2+z-z^2}$$

   in powers of $z+1$. [Notice that you need to consider all situations for the domains in the whole complex plane.]