Complex Analysis Exam-2 Solution

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1.

(a)

$$z=\cos w=\frac{e^{iw}+e^{-iw}}{2}\Rightarrow e^{2iw}-2ze^{iw}+1=0$$

$$\begin{aligned}\Rightarrow e^{iw} &= \frac{2z+(4z^2-4)^{1/2}}{2}\\ &= z+(z^2-1)^{1/2}\\ \Rightarrow w &= -i\log[z+(z^2-1)^{1/2}]\end{aligned}$$

(b)

$$\frac{d}{dz}(\cos^{-1}z)=\frac{d}{dz}\left\{-i\log\left[z+(z^2-1)^{1/2}\right]\right\}$$

$$=-i\frac{1+z(z^2-1)^{-1/2}}{z+(z^2-1)^{1/2}}$$

$$=\frac{-i}{(z^2-1)^{1/2}}\cdot\frac{(z^2-1)^{1/2}+z}{z+(z^2-1)^{1/2}}$$

$$=\frac{-1}{(1-z^2)^{1/2}},\quad z\neq\pm1$$

(c)

$$z=\cos^{-1}(2i)=-i\log\left[2i+(-5)^{1/2}\right]=-i\log[(2\pm\sqrt{5})i]$$

$$=-i\mathrm{Log}(2+\sqrt{5})+\frac{\pi}{2}+2k\pi \text{ and } i\mathrm{Log}(2+\sqrt{5})-\frac{\pi}{2}+2k\pi.$$

2.

$z=e^{i\theta},\ 0\le\theta\le 2\pi$ ∴ $dz=ie^{i\theta}d\theta$

$$\oint_{|z|=1}\bar{z}\,dz=\int_0^{2\pi}e^{-i\theta}\cdot ie^{i\theta}d\theta$$

$$=\int_0^{2\pi}i\,d\theta=2\pi i$$

$\oint_{|z|=1}\frac{1}{z}dz=2\pi i$ ($|z|=1$ encloses $z_0=0$)

$\Rightarrow \oint_{|z|=1}\bar{z}\,dz=\oint_{|z|=1}\frac{1}{z}dz$ ※

3.

The contour C has a parametrization $z(t)=t+(1-t^2)i,\ 0\le t\le1$. Thus, $z'(t)=1-2ti$.

$$\int_C (z^2 - z + 2)dz = \int_0^1 \left((t + (1-t^2) - t - (1-t^2)i + 2 \right)(1 - 2ti)dt$$
$$= \int_0^1 \left(1 - 3t + 7t^2 + 2t^3 - 5t^4 - i1 - 3t^2 + 8t^3 - 2t^5) \right) dt$$
$$= \left[t - \frac{3t^2}{2} + \frac{7t^3}{3} + \frac{t^4}{2} - t^5 - i\left(t - t^3 + 2t^4 - \frac{t^6}{3} \right) \right|_0^1$$
$$= \frac{4}{3} - \frac{5}{3}i.$$

4.

$$\int_C \frac{1}{z^2+1}dz$$

$$|z^2+1| = |z+i||z-i|$$

on the given path:

$$|z+i| > |3+i| = \sqrt{10}$$
$$|z-i| > |3| = 3$$
$$\Rightarrow |z+i|\,|z-i| > 3\sqrt{10} \Rightarrow |z^2+1| > 3\sqrt{10} \Rightarrow \left(\frac{1}{z^2+1} \right) < \frac{1}{3\sqrt{10}} = M$$
$$L = 1$$
$$\Rightarrow \left| \int_C \frac{dz}{z^2+1} \right| < LM = \frac{1}{3\sqrt{10}}$$

5.

$$\frac{1}{z^2(z+2i)} = \frac{A}{z^2} + \frac{B}{z} + \frac{C}{z+2i}$$
$$A = \frac{1}{2i}, \qquad C = \frac{1}{(-2i)^2} = \frac{1}{-4} = -\frac{1}{4}$$
$$f(i) = \frac{1}{-1(i+2i)} = \frac{A}{-1} + \frac{B}{i} + \frac{C}{i+2i}$$
$$\frac{1}{-3i} = \frac{1}{-2i} + \frac{B}{i} - \frac{1}{12i}$$
$$\frac{i}{3} - \frac{i}{2} - \frac{i}{12} = -Bi = \frac{-1}{4}i \Rightarrow B = \frac{1}{4}$$
$$\Rightarrow \frac{1}{z^2(z+2i)} = \frac{\frac{1}{2i}}{z^2} + \frac{\frac{1}{4}}{z} - \frac{\frac{1}{4}}{(z+2i)}$$
$$\oint_C f(z)dz = \oint_C \frac{\frac{1}{2}i}{z^2}dz + \oint_C \frac{\frac{1}{4}}{z}dz - \oint_C \frac{\frac{1}{4}}{z+2i}dz$$

$$\oint_C \frac{\frac{1}{2}i}{z^2}dz \text{ is in the form of } \oint_C \frac{a}{(z-z_0)^n}dz \qquad z_0=0, n=2\neq 1$$

$$\Rightarrow \oint_C \frac{\frac{1}{2}i}{z^2}dz = 0$$

$$\oint_C \frac{\frac{1}{4}}{z}dz = \frac{1}{4}(2\pi i) = \frac{\pi i}{2}$$

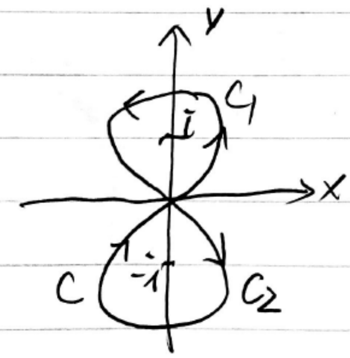
$$\oint_C \frac{\frac{1}{4}}{z+2i}dz = 0 \text{ since } \frac{\frac{1}{4}}{z+2i} \text{ is analytic inside } C$$

$$\Rightarrow \oint_C f(z)dz = \frac{\pi i}{2}$$

6.

$$\oint_C \frac{8z-3}{z^2-z}dz = \oint_{C_1}\frac{8z-3}{z^2-z}dz + \oint_{C_2}\frac{8z-3}{z^2-z}dz$$
$$= -\oint_{-C_1}\frac{8z-3}{z^2-z}dz + \oint_{C_2}\frac{8z-3}{z^2-z}dz$$
$$= -3(0) - 5(2\pi i) + 3(2\pi i) + 5(0) = -4\pi i.$$

7.



$$\int_C \frac{e^{iz}}{(z^2+1)^2}dz$$

$$f(z) = \frac{e^{iz}}{(z^2+1)^2} = \frac{e^{iz}}{(z+i)^2(z-i)^2}$$

$$\int_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$$

$$\oint_{C_1} f(z)dz = \oint_{C_1}\frac{e^{iz}/(z+i)^2}{(z-i)^2}dz$$
$$= 2\pi i\cdot\left(\frac{e^{iz}}{(z+i)^2}\right)'\Bigg|_{z=i}$$
$$= 2\pi i\cdot\frac{ie^{iz}(z+i)^2-2e^{iz}(z+i)}{(z+i)^4}\Bigg|_{z=i}$$
$$= 2\pi i\cdot\frac{ie^{-1}(2i)^2-2e^{-1}(2i)}{(2i)^4}$$
$$= 2\pi i\cdot\frac{-4ie^{-1}-4ie^{-1}}{2^4}.$$

$$= 2\pi i \cdot \frac{-8ie^{-1}}{2^4} = \pi/e$$

$$\oint_{C_2} f(z)\,dz = \oint_{C_2} \frac{e^{iz}/(z-i)^2}{(z+i)^2}\,dz$$

$$= 2\pi i \cdot \left(\frac{e^{iz}}{(z-i)^2} \right)' \Big|_{z=-i}$$

$$= 2\pi i \cdot \frac{ie^{iz}(z-i)^2 - 2e^{iz}(z-i)}{(z-i)^4} \Big|_{z=-i}$$

$$= 2\pi i \cdot \frac{ie \cdot (2i)^2 + 4ie}{(-2i)^4}$$

$$= 2\pi i \cdot 0 = 0$$

$$\int_C f(z)\,dz = \pi/e \quad \#$$