Complex Analysis Exam-3 Solution

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1.

(a) Ans: 2

Consider the power series $\sum_{k=1}^{\infty} \frac{(1)^k}{k2^k}(z-1-i)^k$ about $(z-1-i)$. With the identification

$a_n = (-1)^n / (n2^n)$ we have

$$\lim_{n\to\infty}\left|\frac{\frac{(1)^{n+1}}{(n+1)2^{n+1}}}{\frac{(-1)^n}{n2^n}}\right| = \lim_{n\to\infty}\left|\frac{-n}{(n+12}\right| = \frac{1}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{1}{2}.$$

Hence by (12) of Section 6.1 with $L = 1/2$, we see that the radius of convergence is $R = 2$. The circle of convergence is $|z-1-i| = 2$.

(b) Ans: 25

Consider the power series $\sum_{k=1}^{\infty} \frac{(z-4-3i)^k}{5^{2k}}$ about $(z-4-3i)$. With the identification

$a_n = \frac{1}{5^{2n}}$ we have

$$\lim_{n\to\infty}\left|\frac{\frac{1}{5^{2n+2}}}{\frac{1}{5^{2n}}}\right| = \lim_{n\to\infty}\left|\frac{1}{5^2}\right| = \frac{1}{25}.$$

Hence by (12) of Section 6.1 with $L = 1/25$, we see that the radius of convergence is $R = 25$. The circle of convergence is $|z-4-3i| = 25$.

(c) Ans: 1/2

$a_n = \frac{(2n)!}{(n+2)(n!)^2}$ for $(z-i)^{2n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!/(n+3)[(n+1)!]^2}{(2n)!/(n+2)(n!)^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+2}{n+3}\right|\cdot\left|\frac{n!\cdot n!\,(2n+2)!}{(n+1)!\,(n+1)!\,(2n)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)}{(n+1)^2}\right| = 4$$

radius of convergence for $(z-i)^{2n}$ is $\frac{1}{4}$

$\therefore |z-i| < \frac{1}{2}$ for convergence.

2.

(a) Ans: diverges

$$\left\{\frac{n(1+i^n)}{n+1}\right\}$$

$$z_n = \frac{n(1+i^n)}{n+1}$$

$$\frac{n}{n+1} \to 1 \text{ as } n \to \infty$$

But $1+i^n$ alternates between 0, 1 – i, 2, 1 + i which is why the sequence is divergent

(b) Ans: converges to $1+i\pi$

$$\{e^{1/n} + 2(\tan^{-1} n)i\}$$

$$z_n = e^{1/n} + 2(\tan^{-1} n)i$$

$$\mathrm{Re}(z_n) = e^{1/n} \to 1$$

$$\mathrm{Im}(z_n) = 2(\tan^{-1} n) \to 2\times\frac{\pi}{2} = \pi$$

as $n \to \infty$

Thus z_n is convergent and converges to $1+i\pi$

3.

(a) Ans: $2\pi i$

$$\oint_{|z|=1} \frac{f(z)}{z^4} dz = \frac{2\pi i}{3!} f^{(3)}(0) = \frac{2\pi i(3!a_3)}{3!} = 2\pi i$$

(b)Ans: $\frac{2\pi i}{3}$

$$f(z) = \sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n = \frac{1}{3}z + \frac{8}{9}z^2 + z^3 + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\cdots$$

$$\frac{1}{z^3} f(z)\sin z = \frac{1}{3z} + \frac{8}{9} + \cdots\cdots$$

$$\oint_{|z|=1} \frac{f(z)\sin z}{z^3} dz = \frac{2\pi i}{3} \#$$

4.

$$f(z)=\frac{3-i}{1-i+z}=\frac{3-i}{z-(4+2i)+4+2i+1-i}$$

$$=\frac{3-i}{[z-(4+2i)]+(5-3i)}$$

$$=\frac{3-i}{5-3i}\cdot\frac{1}{1+\frac{z-(4+2i)}{5-3i}}$$

$f(z)$ converges for $\left|\frac{z-(4+2i)}{5-3i}\right|<1$, i.e.

$$|z-(4+2i)|<|5-3i|=\sqrt{5^2+3^2}=\sqrt{34}$$

5. Ans: $z+\frac{z^4}{4}+\frac{z^7}{14}$

$$\int_0^z e^{\tau^3}d\tau=\int_0^z\sum_{k=0}^{\infty}\frac{\tau^{3k}}{k!}d\tau$$

$$=\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\int_0^z\tau^{3k}d\tau$$

$$=\sum_{k=0}^{\infty}\frac{z^{3k+1}}{k!(3k+1)}$$

$$=z+\frac{z^4}{4}+\frac{z^7}{14}+\ldots$$

6.

$$f(z)=\frac{z}{(1-z)^3}$$

For $|z|<1$

$$\frac{1}{1-z}=1+z+z^2+z^3+z^4+\ldots$$

$$\frac{d}{dz}\frac{1}{1-z}=1+2z+3z^2+4z^3+\ldots$$

$$\frac{1}{(1-z)^2}=1+2z+3z^2+4z^3+\ldots$$

$$\frac{d}{dz}\frac{1}{(1-z)^2}=z+6z+12z^2+\ldots$$

$$\frac{2}{(1-z)^3}=2+6z+12z^2+\ldots$$

$$\frac{2}{(1-z)^3} = 2 + 6z + 12z^2 + \ldots$$

$$\frac{2}{(1-z)^3} = \sum_{k=2}^{\infty} k(k-1)z^{k-2}$$

$$\frac{z}{(1-z)^3} = \frac{z}{2}\sum_{k=2}^{\infty} k(k-1)z^{k-2}$$

$$= \frac{1}{2}\sum_{k=2}^{\infty} k(k-1)z^{k-1}$$

$$\frac{z}{(1-z)^3} = \frac{1}{2}\sum_{k=1}^{\infty} (K+1)kz^{k} \qquad R=1$$

7.

$$f(z) = \frac{1+z}{1-z}, \qquad z_0 = i$$

$$\frac{1}{z-1} = \frac{1}{1-z+i-i} = \frac{1}{1-i}\frac{1}{1-\left(\frac{z-i}{1-i}\right)} = \frac{1}{1-i}\sum_{k=0}^{\infty}\left(\frac{z-i}{1-i}\right)^k$$

$$\frac{1}{1-z} = \sum_{k=0}^{\infty}\frac{(z-i)^k}{(1-i)^{k+1}}$$

$$f(z) = \frac{1+z}{1-z} = \frac{1+z+i-i}{1-z}\frac{(1+i)}{1-z} + \frac{z-i}{1-z}$$

$$= (1+i)\sum_{k=0}^{\infty}\frac{(z-i)^k}{(1-i)^{k+1}} + (z-i)\sum_{k=0}^{\infty}\frac{(z-i)^k}{(1-i)^{k+1}}$$

$$= \sum_{k=0}^{\infty}\frac{1+i}{1-i}\cdot\frac{(z-i)^k}{(1-i)^k} + \sum_{k=0}^{\infty}\frac{(z-i)^{k+1}}{(1-i)^{k+1}}$$

$$= \sum_{k=0}^{\infty} i\frac{(z-i)^k}{(1-i)^k} + \sum_{k=0}^{\infty}\frac{(z-i)^k}{(1-i)^k}$$

$$= i + \sum_{k=1}^{\infty} i\frac{(z-i)^k}{(1-i)^k} + \sum_{k=1}^{\infty}\frac{(z-i)^k}{(1-i)^k}$$

$$f(z) = i + \sum_{k=1}^{\infty}(1+i)\frac{(z-i)^k}{(1-i)^k}$$

The series converges for $\left|\frac{z-i}{1-i}\right| < 1$, i.e., $|z-i| < \sqrt{2}$.

Thus $R = \sqrt{2}$

8.

(a)

$$f(z) = \frac{1}{(z-1)^2(z-3)} = \frac{1}{(z-1)^2}\ \frac{1}{-2+(z-1)} = \frac{-1}{2(z-1)^2}\ \frac{1}{1-\dfrac{z-1}{2}}$$

$$f(z) = \frac{-1}{2(z-1)^2}\left[1+\frac{z-1}{2}+\frac{(z-1)^2}{2^2}+\frac{(z-1)^3}{2^3}+\cdots\right]$$

$$= -\frac{1}{2(z-1)^2}-\frac{1}{4(z-1)}-\frac{1}{8}-\frac{1}{16}(z-1)-\cdots.$$

(b)

$$f(z) = -\frac{1}{z}+\frac{1}{z-1} = f_1(z)+f_2(z).$$

$$f_1(z) = -\frac{1}{z} = -\frac{1}{2+z-2}$$

$$= -\frac{1}{2}\ \frac{1}{1+\dfrac{z-2}{2}}$$

$$= -\frac{1}{2}\left[1-\frac{z-2}{2}+\frac{(z-2)^2}{2^2}-\frac{(z-2)^3}{2^3}+\cdots\right]$$

$$= -\frac{1}{2}+\frac{z-2}{2^2}-\frac{(z-2)^2}{2^3}+\frac{(z-2)^3}{2^4}-\cdots$$

$$f_2(z) = \frac{1}{z-1} = \frac{1}{1+z-2} = \frac{1}{z-2}\ \frac{1}{1+\dfrac{1}{z-2}}$$

$$= \frac{1}{z-2}\left[1-\frac{1}{z-2}+\frac{1}{(z-2)^2}-\frac{1}{(z-2)^3}+\cdots\right]$$

$$= \frac{1}{z-2}-\frac{1}{(z-2)^2}+\frac{1}{(z-2)^3}-\frac{1}{(z-2)^4}+\cdots$$

$$f(z) = \cdots-\frac{1}{(z-2)^4}+\frac{1}{(z-2)^3}-\frac{1}{(z-2)^2}+\frac{1}{z-2}-\frac{1}{2}+\frac{z-2}{2^2}-\frac{(z-2)^2}{2^3}+\frac{(z-2)^3}{2^4}-\cdots$$