Complex Analysis Exam-4 Solution

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1.

Write $f(z) = (z - z_0)^m h(z)$

$$\frac{f'(z)}{f(z)} = \frac{m(z - z_0)^{m-1} h(z) + (z - z_0)^m h'(z)}{(z - z_0)^m h(z)}$$

$$= \frac{m}{z - z_0} + \frac{h'(z)}{h(z)}, \text{ with } \frac{h'(z)}{h(z)} \text{ analytic at } z_0.$$

Thus $\frac{f'(z)}{f(z)}$ has a simple pole at z_0 with residue m.

Then, we have

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = m \text{ for } C \text{ enclosing } z_0.$$

2.

(a)

$$\text{Res}(0) = \lim_{z \to 0} \frac{1}{1!} \frac{d}{dz} \left[z^2 f(z) \right] = \lim_{z \to 0} \frac{d}{dz} \left[\frac{\cos z}{(z - \pi)^3} \right]$$

$$= \lim_{z \to 0} \left[\frac{-(z - \pi) \sin z - 3 \cos z}{(z - \pi)^4} \right] = \frac{-3}{\pi^4},$$

$$\text{Res}(\pi) = \lim_{z \to \pi} \frac{1}{2!} \frac{d^2}{dz^2} \left[(z - \pi)^3 f(z) \right] = \lim_{z \to \pi} \frac{1}{2} \frac{d^2}{dz^2} \left[\frac{\cos z}{z^2} \right]$$

$$= \lim_{z \to \pi} \frac{1}{2} \left[\frac{(6 - z^2) \cos z + 4z \sin z}{z^4} \right] = \frac{-(6 - \pi^2)}{2\pi^4}.$$

(b)

$$\text{Res}(n\pi) = \left. \frac{z - 1}{\frac{d}{dz}(\sin z)} \right|_{n\pi} = (-1)^n (n\pi - 1)$$

***n*: integers**

3.

$$e^{1/z} \sin(1/z) = \left(1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots \right) \left(\frac{1}{z} - \frac{1}{6z^3} + \cdots \right) = \left(\frac{1}{z} + \cdots \right)$$

$$\oint_{|z|=1} e^{1/z} \sin(1/z) dz = 2\pi i \ \text{Res}(0) = 2\pi i(1) = 2\pi i$$

4.

(a)

$$I = \text{Im}\left[2\pi i \text{ Res}\left(\frac{ze^{iz}}{z^2-2z+10}; 1+3i\right)\right] = \text{Im}\left[2\pi i \frac{(1+3i)e^{i(1+3i)}}{6i}\right]$$

$$= \frac{\pi}{3e^3}(3\cos 1 + \sin 1)$$

(b)

$$f(z) = \frac{1}{(z^2+1)(z^2+9)} = \frac{1}{(z-i)(z+i)(z-3i)(z+3i)}$$

has two simple poles in the upper half-plane at $z_1 = i$ and $z_2 = 3i$.

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+9)}dx = \text{Re}\left(2\pi i\left[\text{Res}\left(f(z)e^{iz}, i\right) + \text{Res}\left(f(z)e^{iz}, 3i\right)\right]\right).$$

$$\begin{aligned}\text{Res}\left(f(z)e^{iz}, i\right) &= \lim_{z\to i}\left[(z-i)\frac{e^{iz}}{(z-i)(z+i)(z-3i)(z+3i)}\right] \\ &= \lim_{z\to i}\left[\frac{e^{iz}}{(z+i)(z-3i)(z+3i)}\right] \\ &= -\frac{1}{16e}i\end{aligned}$$

$$\begin{aligned}\text{Res}\left(f(z)e^{iz}, 3i\right) &= \lim_{z\to 3i}\left[(z-3i)\frac{e^{iz}}{(z-i)(z+i)(z-3i)(z+3i)}\right] \\ &= \lim_{z\to 3i}\left[\frac{e^{iz}}{(z-i)(z+i)(z+3i)}\right] \\ &= -\frac{1}{48e^3}i\end{aligned}$$

Therefore,

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+9)}dx = \text{Re}\left[2\pi i\left(-\frac{1}{16e}i + \frac{1}{48e^3}i\right)\right] = \frac{\pi}{8}\left(\frac{1}{e} - \frac{1}{3e^3}\right).$$

(c)

$$I = \int_C \frac{\left[\frac{1}{2i}\left(z - \frac{1}{z}\right)\right]^2}{5 + 4\left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]} \frac{dz}{iz}$$

which after some algebra reduces to

$$I = -\frac{1}{4i} \int_C \frac{\left(z^2 - 1\right)^2}{z^2\left(2z^2 + 5z + 2\right)} dz.$$

Clearly the integrand

$$g(z) := \frac{\left(z^2 - 1\right)^2}{z^2\left(2z^2 + 5z + 2\right)} = \frac{\left(z^2 - 1\right)^2}{2z^2\left(z + \frac{1}{2}\right)(z + 2)}$$

has simple poles at $z = -\frac{1}{2}$ and $z = -2$ and has a pole of order 2 at the origin However, only $-\frac{1}{2}$ and 0 lie inside the unit circle C, so that

$$I = -\frac{1}{4i} \cdot 2\pi i \left[\operatorname{Res}\left(g; -\frac{1}{2}\right) + \operatorname{Res}(g; 0)\right].$$

Utilizing the formulas of the preceding section we find

$$\operatorname{Res}\left(g; -\frac{1}{2}\right) = \lim_{z \to -1/2} \left(z + \frac{1}{2}\right) g(z) = \lim_{z \to -1/2} \frac{\left(z^2 - 1\right)^2}{2z^2(z + 2)} = \frac{3}{4},$$

and

$$\operatorname{Res}(g; 0) = \lim_{z \to 0} \frac{1}{1!} \frac{d}{dz}\left[z^2 g(z)\right] = \lim_{z \to 0} \frac{d}{dz}\left[\frac{\left(z^2 - 1\right)^2}{2z^2 + 5z + 2}\right]$$

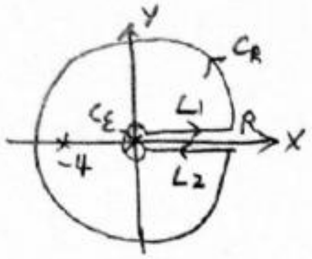
$$= \left.\frac{\left(2z^2 + 5z + 2\right) \cdot 2\left(z^2 - 1\right) 2z - \left(z^2 - 1\right)^2 (4z + 5)}{\left(2z^2 + 5z + 2\right)^2}\right|_{z=0}$$

$$= \frac{-5}{4}.$$

Hence

$$I = \frac{-1}{4i} 2\pi i \left[\frac{3}{4} - \frac{5}{4}\right] = \frac{\pi}{4}. \quad \blacksquare$$

(d)



$C = C_R + C_\varepsilon + L_1 + L_2$, $R \to \infty$, $\varepsilon \to 0$

$$f(z) = \frac{z^{-1/2}}{z+4}$$

$z = re^{i\theta}$, $0<\theta<2\pi$ (f_0 branch)

$$\oint_C f(z)\,dz = \int_{C_R + C_\varepsilon + L_1 + L_2} f(z)\,dz$$

① C_R : $z = Re^{i\theta}$, $R \to \infty$. $dz = Rie^{i\theta}d\theta$

$$\int_{C_R} f(z)\,dz \le \int_{C_R} |f(z)|\,|dz| = \lim_{R\to\infty} \int_0^{2\pi} \frac{R^{-\frac{1}{2}}|e^{i\frac{\theta}{2}}|}{|Re^{i\theta}+4|} R\,d\theta$$

$$\doteq \lim_{R\to\infty} 2\pi R^{-\frac{1}{2}} = 0$$

② C_ε : $z = \varepsilon e^{i\theta}$, $\varepsilon \to 0$ $dz = \varepsilon i e^{i\theta} d\theta$

$$\int_{C_\varepsilon} f(z)\,dz \le \int_{C_\varepsilon} |f(z)|\,|dz| = \lim_{\varepsilon\to 0} \int_0^{2\pi} \frac{\varepsilon^{-\frac{1}{2}}|e^{i\frac{\theta}{2}}|}{|\varepsilon e^{i\theta}+4|} \varepsilon\,d\theta$$

$$= \lim_{\varepsilon \to 0} \frac{1}{4} \cdot 2\pi \cdot \varepsilon^{\frac{1}{2}} = 0$$

③ L_1 : $z = x$, $dz = dx$

$$\int_{L_1} f(z)\,dz = \int_0^\infty \frac{x^{-\frac{1}{2}}}{x+4}\,dx$$

④ L_2 : $z = xe^{i2\pi}$, $dz = dx$.

$$\int_{L_2} f(z)\,dz = \int_\infty^0 \frac{x^{-\frac{1}{2}}e^{-i\pi}}{x+4}\,dx = -\int_\infty^0 \frac{x^{-\frac{1}{2}}}{x+4}\,dx = \int_0^\infty \frac{x^{-\frac{1}{2}}}{x+4}\,dx$$

$$\oint_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=-4} f(z) = 2\pi i \left(z^{-\frac{1}{2}}\right)_{z=-4} = 2\pi i\, e^{-\frac{1}{2}(\ln|-4| + i\cdot \arg(-4))}$$

$$= 2\pi i \cdot e^{-\frac{1}{2}\ln 4} \cdot e^{-i\pi/2} = 2\pi i \cdot \left(\frac{-1}{2} i \sin\frac{\pi}{2}\right) = \pi$$

$$\Rightarrow \quad 2\int_0^\infty \frac{x^{-\frac{1}{2}}}{x+4}\,dx = \pi$$

$$\therefore \int_0^\infty \frac{dx}{\sqrt{x}\,(x+4)} = \frac{\pi}{2}$$