#1. $\left(\frac{1+i\tan\theta}{1-i\tan\theta}\right)^n = \left(\frac{\cos\theta+i\sin\theta}{\cos\theta-i\sin\theta}\right)^n = \frac{\cos n\theta+i\sin n\theta}{\cos n\theta-i\sin n\theta} = \frac{1+i\tan n\theta}{1-i\tan n\theta}$

#2. Let $z=x+iy$, $\Delta z=\Delta x+i\Delta y$

① differentiate $f(z)$ along the x-axis

$$\frac{d}{dz}f(z) = \lim_{\substack{\Delta x\to 0\\ \Delta y=0}} \frac{|x+\Delta x+iy|-|x+iy|}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{\sqrt{(x+\Delta x)^2+y^2}-\sqrt{x^2+y^2}}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{2\cdot\Delta x\cdot x+\Delta x^2}{\Delta x\left(\sqrt{(x+\Delta x)^2+y^2}+\sqrt{x^2+y^2}\right)}$$

$$= \frac{x}{\sqrt{x^2+y^2}}, \text{ if } z\neq 0$$

② differentiate $f(z)$ along y-axis

$$\frac{d}{dz}f(z) = \lim_{\substack{\Delta y\to 0\\ \Delta x=0}} \frac{|x+iy+i\Delta y|-|x+iy|}{i\Delta y}$$

$$= \lim_{\Delta y\to 0} \frac{\sqrt{x^2+(y+\Delta y)^2}-\sqrt{x^2+y^2}}{i\Delta y}$$

$$= \lim_{\Delta y\to 0} \frac{2\cdot\Delta y\cdot y+\Delta y^2}{i\Delta y\cdot\left(\sqrt{x^2+(y+\Delta y)^2}+\sqrt{x^2+y^2}\right)}$$

$$= \frac{y}{i\cdot\sqrt{x^2+y^2}}, \text{ if } z\neq 0.$$

$\Rightarrow$ If $z\neq 0$, $ix=y$. $f(z)$ is not differentiable.

when $z=0$ i.e. $x=0, y=0$.

① along x-axis,

$$\frac{d}{dz}f(z) = \lim_{\Delta x\to 0}\frac{|\Delta x|}{\Delta x} = \pm 1$$

② along y-axis

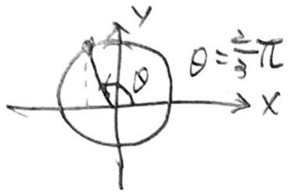
$$\frac{d}{dz}f(z) = \lim_{\Delta y\to 0}\frac{|i\Delta y|}{i\Delta y} = \lim_{\Delta y\to 0}\frac{-i\cdot|\Delta y|}{\Delta y} = \pm i$$

$\Rightarrow f(z)$ is not differentiable.

#3. $z^6+z^3+1=0 \qquad (z^3)^2+(z^3)+1=0$

$z^3=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1}{2}\pm\frac{\sqrt{3}}{2}i$

① $z^3=\frac{-1}{2}+\frac{\sqrt{3}}{2}i=e^{i\frac{2}{3}\pi}$

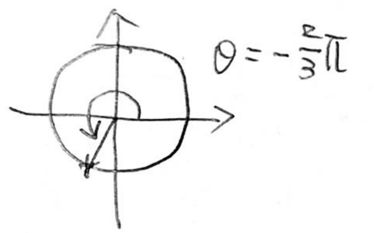


$z=e^{i(\frac{2}{9}\pi+\frac{2n\pi}{3})}, \quad n=0,1,2$

$=e^{i\frac{2}{9}\pi},\ e^{i\frac{8}{9}\pi},\ e^{i\frac{14}{9}\pi}$

$=\begin{cases}\cos 40^\circ+i\sin 40^\circ=0.766+i0.643\\ \cos 160^\circ+i\sin 160^\circ=-0.94+i0.342\\ \cos 280^\circ+i\sin 280^\circ=0.174-i0.985\end{cases}$

② $z^3=\frac{-1}{2}-\frac{\sqrt{3}}{2}i=e^{-i\frac{2}{3}\pi}$



$z=e^{i(-\frac{2}{9}\pi+\frac{2n\pi}{3})}, \quad n=0,1,2.$

$=e^{-i\frac{2}{9}\pi},\ e^{i\frac{4}{9}\pi},\ e^{i\frac{10}{9}\pi}$

$=\begin{cases}\cos(-40^\circ)+i\sin(-40^\circ)=0.766-i0.643\\ \cos(80^\circ)+i\sin(80^\circ)=0.174+i0.985\\ \cos(200^\circ)+i\sin(200^\circ)=-0.94-i0.342\end{cases}$

#4

$f(z)=(x^2+y)+i(y^2-x)$

$u=x^2+y,\ v=y^2-x$

$u_x=2x,\ v_y=2y$

$u_y=1,\ v_x=-1$

By Cauchy-Riemann equations.

$u_x=v_y,\ u_y=-v_x$

$\Rightarrow x=y$ i.e. CR eq can be only satisfied on the line $x=y$, there is no neighboorhood.

$\therefore f$ is nowhere analytic.

#5

v is a harmonic conjugate of u

$u_x = v_y$, $u_y = -v_x$ $\quad u_{xx} + u_{yy} = 0$, $v_{xx} + v_{yy} = 0$

$$\frac{\partial^2}{\partial x^2} uv = \frac{\partial}{\partial x}(u_x v + u v_x)$$
$$= u_{xx} v + 2 u_x v_x + u v_{xx}$$
$$\frac{\partial^2}{\partial y^2} uv = u_{yy} v + 2 u_y v_y + u v_{yy}$$

$\left.\right\} u_x v_x + u_y v_y = 0$.

$$\frac{\partial^2}{\partial x^2} uv + \frac{\partial^2}{\partial y^2} uv = (u_{xx} + u_{yy}) v + 2(u_x v_x + u_y v_y)$$
$$+ u(v_{xx} + v_{yy}) = 0.$$

#6

$f = u + iv$, $u = \frac{y}{x^2+y^2}$

$u_x = v_y$, $u_y = -v_x$

$$\therefore u_x = \frac{\partial}{\partial x}(y \cdot (x^2+y^2)^{-1}) = \frac{-2xy}{(x^2+y^2)^2} = v_y$$

$$\Rightarrow v = \int u_x \, dy + h(x)$$
$$= \frac{x}{x^2+y^2} + h(x)$$

$$v_x = \frac{\partial}{\partial x}(x (x^2+y^2)^{-1}) + h'(x) = \frac{y^2 - x^2}{(x^2+y^2)^2} + h'(x)$$
$$= -u_y = -\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\therefore h'(x) = 0 \quad h(x) = C$, const.

$$f(z) = \frac{y + ix}{x^2+y^2} + C = \frac{i\bar{z}}{|z|^2} + C = \frac{i}{z} + C$$

$/\!/$

#7. $f = u(r,\theta) + i\,v(r,\theta)$, $u = r^2 \sin 2\theta$

$u_r = \frac{1}{r} v_\theta$, $v_r = -\frac{1}{r} u_\theta$

$\Rightarrow v_\theta = r \cdot u_r = r \cdot (2r \sin 2\theta)$

$= 2r^2 \sin 2\theta$

$v = -r^2 \cos 2\theta + h(r)$

$-r \cdot v_r = -r(-2r\cos 2\theta + h'(r))$

$= 2r^2 \cos 2\theta - r h'(r)$

$= u_\theta = 2r^2 \cos 2\theta$

$\Rightarrow r h'(r) = 0$, $h'(r) = 0$, $h(r) = C$.

$\therefore v(r,\theta) = -r^2 \cos 2\theta + C$.

$f(z) = r^2(\sin 2\theta - i\cos 2\theta) + C$

$= -i \cdot r^2(\cos 2\theta + i \sin 2\theta) + C$

$= -i z^2 + C$ ※