2020 Complex Analysis Exam·2

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#1. $\mathrm{Log}(z^2-1)=\frac{\pi i}{2} \Rightarrow z^2-1=e^{\frac{\pi i}{2}}=i$

$z^2=1+i$, $z=(1+i)^{1/2}=2^{\frac{1}{4}}e^{i(\frac{\pi}{4}+2k\pi)\cdot\frac{1}{2}}$, $k=0,1.$

$\Rightarrow z=\underline{2^{\frac{1}{4}}(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}),\ 2^{\frac{1}{4}}(\cos\frac{9\pi}{8}+i\sin\frac{9\pi}{8})}$

#2. (a) $2^{\pi i}=e^{\pi i\ln 2}=e^{\pi i\cdot(\mathrm{Ln}2+i\arg 2)}$

$=e^{\pi i(\mathrm{Ln}2+i\cdot 2k\pi)}$, $k=0,\pm1,\pm2,\cdots$

$=\underline{e^{-2k\pi^2}(\cos\pi\mathrm{Ln}2+i\sin\pi\cdot\mathrm{Ln}2)}$

(b) $(1-i)^{1+i}=(1-i)(1-i)^i=(1-i)\cdot e^{i\ln(1-i)}$

$=(1-i)\cdot e^{i[\mathrm{Ln}\sqrt{2}+i\cdot(\frac{-\pi}{4}+2k\pi)]}$, $k=0,\pm1,\pm2,\cdots$

$=(1-i)\cdot e^{\frac{\pi}{4}-2k\pi}\cdot e^{i\mathrm{Ln}\sqrt{2}}$

$=e^{\frac{1-8k}{4}\pi}\cdot(1-i)\cdot(\cos\mathrm{Ln}\sqrt{2}+i\sin\mathrm{Ln}\sqrt{2})$

$=\underline{e^{\frac{1-8k}{4}\pi}[(\cos\mathrm{Ln}\sqrt{2}+\sin\mathrm{Ln}\sqrt{2})+i(\sin\mathrm{Ln}\sqrt{2}-\cos\mathrm{Ln}\sqrt{2})]}$

(c) $(-1)^{2/3}=e^{\frac{2}{3}\ln(-1)}=e^{\frac{2}{3}\cdot i(\pi+2k\pi)}$ $k=0,\pm1,\pm2,\cdots$

$=\cos(\frac{2}{3}(1+2k)\pi)+i\sin(\frac{2}{3}(1+2k)\pi)$

$=e^{i\frac{2}{3}\pi},\ e^{i2\pi},\ e^{i\frac{4}{3}\pi}$

$=\underline{1,\ \frac{-1\pm\sqrt{3}i}{2}}$

#3.

(a) $f(z) = (1-z)^{-2}$

$f'(z) = 2(1-z)^{-3}$, $f''(z) = 3\cdot 2\cdot (1-z)^{-4}$

$\vdots$

$f^{(n)}(z) = (n+1)!\cdot(1-z)^{-(n+2)}$

$f^{(n)}(0) = (n+1)!$

(b) Let $C: |z| = R$, $0 < R < 1$.

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

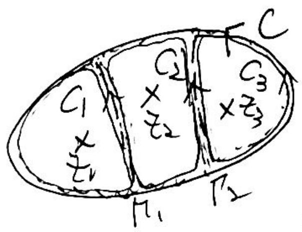
$$f^{(n)}(0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}dz$$

$\because |z| = R$, $|f(z)| \le \frac{1}{(1-|z|)^2} = \frac{1}{(1-R)^2}$

$$|f^{(n)}(0)| \le \frac{n!}{2\pi}\cdot\oint_C \left|\frac{f(z)}{z^{n+1}}\right| dz$$

$$(n+1)! \le \frac{n!}{2\pi}\cdot\frac{2\pi R}{R^{n+1}(1-R)^2} = \frac{n!}{R^n(1-R)^2}$$

#4.



$f(z)$ is analytic, $\because \int_{\Gamma_1} = -\int_{-\Gamma_1}$, $\int_{\Gamma_2} = -\int_{-\Gamma_2}$

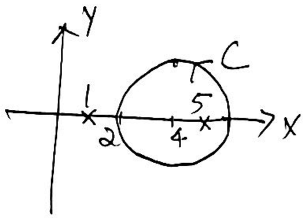
$\oint_C = \oint_{C_1+C_2+C_3}$

$$g(z) = \frac{f(z)}{(z-z_1)(z-z_2)(z-z_3)}$$

$$\because \oint_C g(z)dz = \oint_{C_1}\frac{g(z)(z-z_1)}{z-z_1}dz + \oint_{C_2}\frac{g(z)(z-z_2)}{z-z_2}dz + \oint_{C_3}\frac{g(z)(z-z_3)}{z-z_3}dz$$

$$\therefore \frac{1}{2\pi i}\oint_C g(z)dz = \frac{f(z_1)}{(z_1-z_2)(z_1-z_3)} + \frac{f(z_2)}{(z_2-z_1)(z_2-z_3)} + \frac{f(z_3)}{(z_3-z_1)(z_3-z_2)}$$

#5.



$$\oint_C \frac{\cos z}{(z-1)^3(z-5)^2}\,dz$$

$$= \oint_C \frac{\cos z/(z-1)^3}{(z-5)^2}\,dz$$

$$= 2\pi i \cdot \frac{d}{dz}\left(\frac{\cos z}{(z-1)^3}\right)_{z=5}$$

$$= 2\pi i \cdot \frac{-\sin z \cdot (z-1)^3 - 3\cdot\cos z\cdot(z-1)^2}{(z-1)^6}\Bigg|_{z=5}$$

$$= 2\pi i \cdot \frac{-4\sin 5 - 3\cos 5}{4^4}$$

$$= -\pi i \cdot \frac{4\sin 5 + 3\cos 5}{128}$$

#6.

$C: z = 1 - i + e^{it},\quad 0 \le t \le \pi \quad , dz = i\cdot e^{it}dt$

$$\int_C (|z-1+i|^2 - z)\,dz$$

$$= \int_0^\pi (|e^{it}|^2 - 1 + i - e^{it}) \cdot i e^{it}\,dt$$

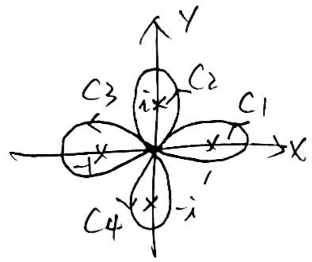
$$= \int_0^\pi (i - e^{it}) \cdot i e^{it}\,dt$$

$$= \int_0^\pi (-1)\cdot e^{it}\,dt - i\int_0^\pi e^{i2t}\,dt$$

$$= i e^{it}\Big|_0^\pi - \frac{1}{2} e^{i2t}\Big|_0^\pi$$

$$= i(e^{i\pi} - 1) - \frac{1}{2}(e^{i2\pi} - 1) = -2i$$

#7.



$$C = C_1 + C_2 + C_3 + C_4$$

$$\oint_C \frac{dz}{z^4-1} = \oint_{C_1+C_2+C_3+C_4} \frac{dz}{z^4-1}$$

$$z^4-1 = (z^2+1)(z^2-1)$$
$$= (z+i)(z-i)(z+1)(z-1)$$

$$\oint_{C_1} \frac{dz}{z^4-1} = 2\pi i \cdot \frac{1}{(1+1)\cdot(1+1)} = \frac{2\pi i}{4}$$

$$\oint_{C_2} \frac{dz}{z^4-1} = 2\pi i \cdot \frac{1}{2i\cdot(-2)} = \frac{2\pi i}{-4i}$$

$$\oint_{C_3} \frac{dz}{z^4-1} = 2\pi i \cdot \frac{1}{2\cdot(-2)} = -\frac{2\pi i}{4}$$

$$\oint_{C_4} \frac{dz}{z^4-1} = 2\pi i \cdot \frac{1}{(-2i)\cdot(-2)} = \frac{2\pi i}{4i}$$

$$\Rightarrow \oint_C \frac{dz}{z^4-1} = 0$$