Complex Analysis 2020 Exam-3

D.C. Chang 2020/12/18.

#1.

(a)

$$a_n = i^n - \frac{1}{n^2}$$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} |i^n| \neq 0$$

$\Rightarrow$ the series diverges.

(b)

By ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)\cdot n^n}{(n+1)^n\cdot(n+1)}$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e} < 1$$

$\Rightarrow$ the series converges.

#2.

(a)

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{z^n/n!}{z^{n+1}/(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{n+1}{z}\right| \to \infty$$

$\Rightarrow$ the series converges for all z. $R=\infty$ #

(b) By ratio test,

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(z+5i)^{2n+2}\cdot(n+2)^2}{(z+5i)^{2n}\cdot(n+1)^2}\right|$$

$$= |(z+5i)^2|$$

the series converges when $|z+5i|<1$ #

(c)

$$a_n = \frac{z^n}{z^n - 3^n} = \frac{1}{1-\left(\frac{3}{z}\right)^n} \to 1 \text{ as } n\to\infty \text{ for } \left|\frac{3}{z}\right|<1$$

$$\to 0 \text{ as } n\to\infty \text{ for } \left|\frac{3}{z}\right|>1$$

$\Rightarrow \sum_{n=0}^{\infty} a_n$ converges when $|z|<3$ #

#3. $f(z)$ has two poles at $z^2+1=0$, i.e. $z=\pm i$

$$|i-(2+5i)|=|-2-4i|=\sqrt{20}=2\sqrt{5}$$

$$|-i-(2+5i)|=|-2-6i|=\sqrt{40}=2\sqrt{10}$$

the radius of convergence is $\underline{2\sqrt{5}}$ #

#4.

$$\sum_{n=-\infty}^{\infty}\frac{z^n}{2^{|n|}}=\sum_{n=0}^{\infty}\frac{z^n}{2^n}+\sum_{n=1}^{\infty}\frac{z^{-n}}{2^n}=\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n+\sum_{n=1}^{\infty}\left(\frac{1}{2z}\right)^n$$

the series converges when $\left|\frac{z}{2}\right|<1$ & $\left|\frac{1}{2z}\right|<1$

$\Rightarrow$ $\underline{\frac{1}{2}<|z|<2}$ #

#5. With Taylor series expansion,

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n=f(0)+f'(0)z+\frac{f''(0)}{2}z^2+\cdots$$

$\because$ $f(0)=f'(0)=0$

$$\therefore f(z)=\frac{f''(0)}{2}z^2+\frac{f'''(0)}{6}z^3+\cdots$$

$$=z^2\cdot\left(\frac{f''(0)}{2}+\frac{f'''(0)}{6}z+\cdots\right)=z^2\cdot g(z)$$

$\because$ $f(z)$ is analytic, $g(z)=\frac{f''(0)}{2}+\frac{f'''(0)}{6}z^2+\cdots$ is analytic.

#6 (a)

$$\frac{1}{1-z}=\frac{1}{1-z+i-i}=\frac{1}{(1-i)-(z-i)}=\frac{1}{1-i}\cdot\frac{1}{1-\frac{z-i}{1-i}}$$

$$=\frac{1}{1-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n \quad , \left|\frac{z-i}{1-i}\right|<1, \text{ ie., } |z-i|<\sqrt{2}$$

$$=\sum_{n=0}^{\infty}\frac{1}{(1-i)^{n+1}}(z-i)^n$$

$$=\frac{1+i}{2}+\frac{i}{2}(z-i)+\frac{i-1}{4}(z-i)^2-\frac{1}{4}(z-i)^4+\cdots$$

(b)

$f(z) = \mathrm{Ln}(1-z)$, $f(0)=0$, $f'(z) = \frac{-1}{1-z}$, $f''(z) = \frac{-1}{(1-z)^2}$

$f^{(n)}(z) = \frac{-(n-1)!}{(1-z)^n}$

$f(z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots$

$f(z) = \ln(1+z)$, $f(0)=0$, $f'(z) = \frac{1}{1+z}$, $f''(z) = \frac{-1}{(1+z)^2}$

$f^{(n)}(z) = \frac{(-1)^{n-1} \cdot (n-1)!}{(1+z)^n}$

$f(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$

$$\mathrm{Log}\left(\frac{1-z}{1+z}\right) = \mathrm{Log}(1-z) - \mathrm{Log}(1+z)$$
$$= \left(-z - \cancel{\frac{z^2}{2}} - \frac{z^3}{3} - \cancel{\frac{z^4}{4}} - \cdots\right) - \left(z - \cancel{\frac{z^2}{2}} + \frac{z^3}{3} - \cancel{\frac{z^4}{4}} + \cdots\right)$$
$$= -2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \cdots\right)$$

#7 $f(z) = \frac{z}{(z+1)^2(z-2)} = \frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{z-2}$

$$z = A(z+1)(z-2) + B(z-2) + C(z+1)^2$$
$$= (A+C)z^2 + (-A+B+2C)z + (-2A-2B+C)$$

$$\begin{cases} A+C=0 \\ -A+B+2C=1 \\ -2A-2B+C=0 \end{cases} \Rightarrow A = -\frac{2}{9},\ B = \frac{1}{3},\ C = \frac{2}{9}$$

$\frac{-\frac{2}{9}}{1+z} = -\frac{2}{9}(1 - z + z^2 - z^3 + \cdots)$ $\quad |z|<1$

$\frac{\frac{1}{3}}{(1+z)^2} = \frac{1}{3}\cdot(-1)\cdot\frac{d}{dz}\left(\frac{1}{1+z}\right) = -\frac{1}{3}(-1 + 2z - 3z^2 + 4z^3 - \cdots)$

$= \frac{1}{3}(1 - 2z + 3z^2 - 4z^3 + \cdots)$ $\quad |z|<1$

$\frac{\frac{2}{9}}{z-2} = -\frac{1}{9}\cdot\frac{1}{1-z/2} = -\frac{1}{9}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right)$ $\quad |z|<2$

$$f(z) = \left(-\frac{2}{9}+\frac{1}{3}-\frac{1}{9}\right)+\left(\frac{2}{9}-\frac{2}{3}-\frac{1}{18}\right)z+\left(-\frac{2}{9}+1-\frac{1}{36}\right)z^2$$
$$+\left(\frac{2}{9}-\frac{4}{3}-\frac{1}{72}\right)z^3+\cdots$$
$$= -\frac{1}{2}z+\frac{3}{4}z^2-\frac{9}{8}z^3+\cdots \quad , |z|<1 \quad \#$$

#8.

$$f(z) = \frac{1}{(z-2)(z-1)^3} = \frac{1}{z-2}\cdot\frac{1}{(1+(z-2))^3} \quad , z\neq 1,2$$

$$\because \frac{d^2}{dz^2}\left(\frac{1}{1+(z-2)}\right) = \frac{2}{[1+(z-2)]^3}$$

$$\therefore \frac{1}{[1+(z-2)]^3} = \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{1}{1+(z-2)}\right)$$

$$\frac{1}{1+(z-2)} = 1-(z-2)+(z-2)^2-(z-2)^3+(z-2)^4-\cdots, |z-2|<1$$

$$\frac{1}{2}\frac{d^2}{dz^2}\left(\frac{1}{1+(z-2)}\right) = \frac{1}{2}\frac{d}{dz}\left(-1+2(z-2)-3(z-2)^2+4(z-2)^3-\cdots\right)$$
$$= 1-3(z-2)+6(z-2)^2-10(z-2)^3+\cdots$$

$$f(z) = \frac{1}{z-2}\left(1-3(z-2)+6(z-2)^2-10(z-2)^3+\cdots\right)$$
$$= \frac{1}{z-2}-3+6(z-2)-10(z-2)^2+\cdots \quad \#$$