Complex Analysis 2020. Final

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#1. Let $z = e^{i\theta}$, $0 \le \theta \le 2\pi$, $dz = i e^{i\theta} d\theta = i z \, d\theta$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2z}\cdot(z^2+1) \quad , C: |z|=1$$

$$\int_0^{2\pi} \frac{d\theta}{10 - 6\cos\theta} = \oint_C \frac{1}{10 - \frac{3}{z}(z^2+1)} \cdot \frac{1}{iz} dz$$

$$= \oint_C \frac{-i}{10z - 3z^2 - 3} dz = \oint_C \frac{i}{3z^2 - 10z + 3} dz$$

$$f(z) = \frac{1}{3z^2 - 10z + 3} = \frac{1}{(z-3)(3z-1)} \quad , \text{only } z = \tfrac{1}{3} \text{ inside } C.$$

$$i \oint_C f(z) dz = i \cdot 2\pi i \cdot \operatorname{Res}(f(z), \tfrac{1}{3})$$

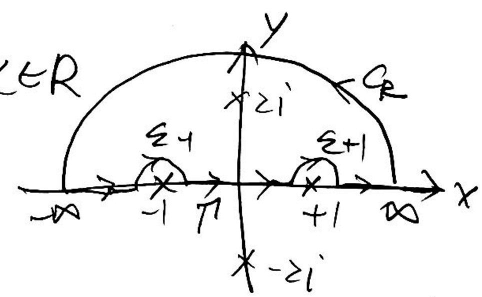
$$= -2\pi \cdot \frac{1}{3} \cdot \frac{1}{z-3}\Big|_{z=\frac{1}{3}} = -2\pi \cdot \frac{1}{3} \cdot \frac{3}{-8} = \frac{\pi}{4} \quad \#$$

#2.

$$I = \int_0^\infty \frac{x \sin x}{(x^2-1)(x^2+4)} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{(x^2-1)(x^2+4)} dx \quad , x\sin x \text{ is even}$$

$$= \frac{1}{2} \operatorname{Im}\left\{ \int_{-\infty}^{\infty} \frac{x e^{ix}}{(x^2-1)(x^2+4)} dx \right\} , \ x \in \mathbb{R}$$

Let $f(z) = \dfrac{z e^{iz}}{(z^2-1)(z^2+4)}$



$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x e^{ix}}{(x^2-1)(x^2+4)} dx = \int_{\Pi_\infty} f(z) dz$$

$C = C_R + \Pi_\infty + \varepsilon_{-1} + \varepsilon_{+1}$

$\int_{C_R} f(z) dz = 0.$

$$\oint_C = \int_{C_R} + \int_{\Pi_\infty} + \int_{\varepsilon_{-1}} + \int_{\varepsilon_{+1}}$$

$$\int_{\Pi_\infty} = 2\pi i \cdot \operatorname{Res}(f(z), 2i) + \pi i \left[\operatorname{Res}(f(z), -1) + \operatorname{Res}(f(z), 1) \right].$$

$$\operatorname{Res}(f(z), 2i) = \frac{z e^{iz}}{(z^2-1)(z+2i)}\Big|_{z=2i} = \frac{2i \cdot e^{-2}}{(-4-1)\cdot 4i} = -\frac{e^{-2}}{10}$$

$$\operatorname{Res}(f(z), -1) = \frac{z e^{iz}}{(z-1)(z^2+4)}\Big|_{z=-1} = \frac{-e^{-i}}{-2\cdot 5} = \frac{e^{-i}}{10}$$

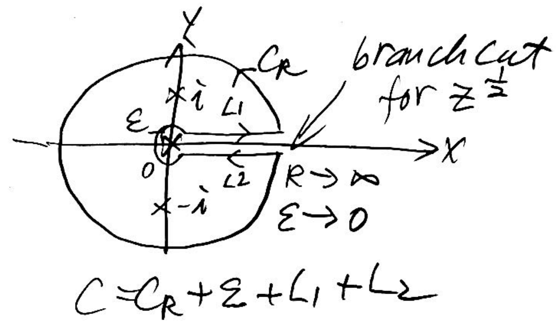
$$\operatorname{Res}(f(z), 1) = \frac{z e^{iz}}{(z+1)(z^2+4)}\Big|_{z=1} = \frac{e^{i}}{2\cdot 5} = \frac{e^{i}}{10}$$

$$\int_{\Gamma_\infty} = 2\pi i\cdot\left(-\frac{e^{-2}}{10}\right) + \pi i\cdot\left(\frac{e^{-i}}{10}+\frac{e^{i}}{10}\right)$$

$$= -\frac{\pi i}{5}e^{-2} + \frac{\pi i}{5}\cos 1 = \frac{\pi i}{5}(\cos 1 - e^{-2})$$

$$I = \frac{1}{2}\,\mathrm{Im}\left\{\frac{\pi i}{5}(\cos 1 - e^{-2})\right\} = \underline{\frac{\pi}{10}(\cos 1 - e^{-2})}$$ ※

#3. $I = \int_0^\infty \frac{x^{\frac{1}{2}}}{x^2+1}\,dx$



Let $f(z) = \frac{z^{\frac{1}{2}}}{z^2+1}$

$$\oint_C = \int_{C_R} + \int_\varepsilon + \int_{L_1} + \int_{L_2}$$

$C = C_R + \varepsilon + L_1 + L_2$

$\int_{C_R} f(z)\,dz \to 0$ as $R\to\infty$

$\int_\varepsilon f(z)\,dz \to 0$ as $\varepsilon \to 0$

$L_1: z = x,\ dz = dx$

$L_2: z = x e^{i2\pi},\ dz = dx$

$\int_{L_1} f(z)\,dz = I$

$$\int_{L_2} f(z)\,dz = \int_\infty^0 \frac{x^{\frac{1}{2}}\cdot e^{i\pi}}{x^2+1}\,dx = -e^{i\pi}\cdot I$$

$$\oint_C f(z)\,dz = 2\pi i\left[\mathrm{Res}(f(z), i) + \mathrm{Res}(f(z), -i)\right]$$

$$= 2\pi i\cdot\left(\left.\frac{z^{1/2}}{z+i}\right|_{z=i} + \left.\frac{z^{1/2}}{z-i}\right|_{z=-i}\right)$$

$$= 2\pi i\cdot\frac{1}{2i}\left(i^{\frac{1}{2}} - (-i)^{\frac{1}{2}}\right)$$

$$= \pi\left(e^{\frac{\pi}{4}i} - e^{\frac{3\pi}{4}i}\right)$$

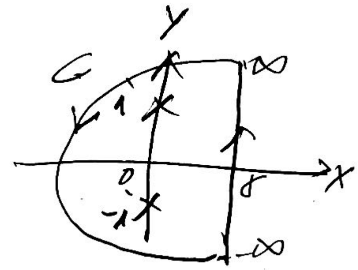
$$= \pi\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i - \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)\right] = \sqrt{2}\pi$$

$$(1 - e^{i\pi})I = \sqrt{2}\pi$$

$$2I = \sqrt{2}\pi, \quad \underline{I = \frac{\pi}{\sqrt{2}}}$$ ※

#4. $\mathcal{L}^{-1}\left\{\frac{1}{s^4+2s^2+1}\right\}$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{zt}}{z^4+2z^2+1}\,dz$$



$\gamma > 0$

$$F(z)=\frac{e^{zt}}{(z^2+1)^2}=\frac{e^{zt}}{(z+i)^2(z-i)^2}$$

$$f(t)=\operatorname{Res}(F(z),i)+\operatorname{Res}(F(z),-i)$$

$$=\frac{d}{dz}\frac{e^{zt}}{(z+i)^2}\bigg|_{z=i}+\frac{d}{dz}\frac{e^{zt}}{(z-i)^2}\bigg|_{z=-i}$$

$$=\frac{te^{zt}(z+i)^2-2e^{zt}(z+i)}{(z+i)^4}\bigg|_{z=i}+\frac{te^{zt}(z-i)^2-2e^{zt}(z-i)}{(z-i)^4}\bigg|_{z=-i}$$

$$=\frac{t(z+i)-2}{(z+i)^3}\cdot e^{zt}\bigg|_{z=i}+\frac{t(z-i)-2}{(z-i)^3}\,e^{zt}\bigg|_{z=-i}$$

$$=2\cdot\frac{it-1}{-8i}\cdot e^{it}-2\cdot\frac{it+1}{8i}\cdot e^{-it}$$

$$=-\frac{1}{4i}\left(it\cdot(e^{it}+e^{-it})-(e^{it}-e^{-it})\right)$$

$$=-\frac{1}{4i}\left(it\cdot 2\cos t-2i\sin t\right)$$

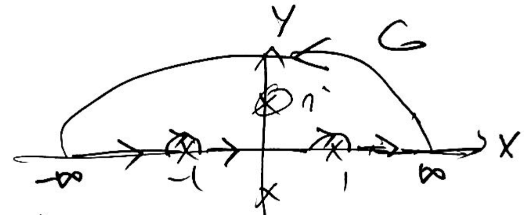
$$=\frac{1}{2}(\sin t-t\cos t)$$ #

#5 $F(w)=\frac{w^2}{\pi(w^4-1)}$

$$f(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{w^2e^{iwt}}{w^4-1}\,dw=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{z^2e^{izt}}{z^4-1}\,dz\quad,\ \bar{f}=\frac{z^2e^{izt}}{z^4-1}$$

$$z^4-1=(z^2-1)(z^2+1)=(z-1)(z+1)(z+i)(z-i)$$

① $t>0$



$$\int_{-\infty}^{\infty}\frac{z^2e^{izt}}{z^4-1}\,dz=2\pi i\,Res(\bar{f},i)+\pi i\left(Res(\bar{f},-1)+Res(\bar{f},+1)\right)$$

$$=2\pi i\cdot\frac{z^2e^{izt}}{(z^2-1)(z+i)}\Big|_{z=i}+\pi i\,\frac{z^2e^{izt}}{(z-1)(z^2+1)}\Big|_{z=-1}$$

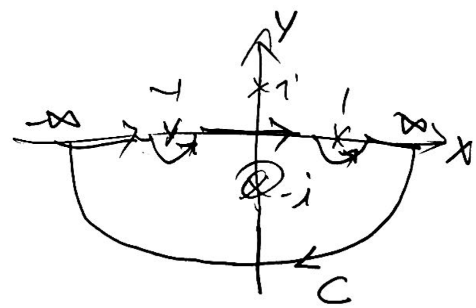
$$+\pi i\,\frac{z^2e^{izt}}{(z+1)(z^2+1)}\Big|_{z=1}$$

$$=2\pi i\cdot\frac{-1\cdot e^{-t}}{-2\cdot 2i}+\pi i\left(\frac{e^{-it}}{-2\cdot 2}+\frac{e^{it}}{2\cdot 2}\right)$$

$$=\frac{\pi}{2}e^{-t}+\frac{\pi i}{4}\cdot 2i\sin t=\frac{\pi}{2}(e^{-t}-\sin t)$$

$$f(t)=\frac{1}{2}(e^{-t}-\sin t)$$

② $t<0$



$$\int_{-\infty}^{\infty}\bar{f}\,dz=-2\pi i\,Res(\bar{f},-i)-\pi i\left(Res(\bar{f},-1)+Res(\bar{f},1)\right)$$

$$=-2\pi i\cdot\frac{z^2e^{izt}}{(z^2-1)(z-i)}\Big|_{z=-i}+\frac{\pi}{2}\sin t$$

$$=\cancel{-2}\pi i\,\frac{\cancel{-1}\cdot e^{t}}{\cancel{-2}\cdot(-2i)}+\frac{\pi}{2}\sin t=\frac{\pi}{2}e^{t}+\frac{\pi}{2}\sin t$$

$$f(t)=\frac{1}{2}(e^{t}+\sin t)$$

$$\Rightarrow\ \underline{f(t)=\frac{1}{2}\left(e^{-|t|}-\sin|t|\right)}\ ,\ t\in\mathbb{R}.$$

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