CO2013: Complex Analysis, Exam-3, Fall 2020

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Notice:
a) Term grading policy: Exam-3×25%.
b) Total 100 points in this exam.
c) Exam Time: 1:00PM−2:50PM, Dec. 18, 2020.

1. (10 pts) Explain whether the following series converge or diverge:

   (a) $\sum_{n=1}^{\infty} (i^n - \frac{1}{n^2})$, (b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

2. (15 pts) Find the domain in which convergence holds for each of the following series of functions:

   (a) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, (b) $\sum_{n=0}^{\infty} (z+5i)^{2n}(n+1)^2$,

   (c) $\sum_{n=1}^{\infty} \frac{z^n}{z^n - 3^n}$.

3. (10 pts) Determine the radius of convergence of the Taylor series of the the function $f(z) = \frac{4+5z}{1+z^2}$ at the center point $z_0 = 2+5i$.

4. (10 pts) Determine the annulus of convergence of the Laurent series $\sum_{n=-\infty}^{\infty} \frac{z^n}{2^{|n|}}$.

5. (10 pts) Assume that $f(z)$ is analytic at the origin and that $f(0) = f'(0) = 0$. Prove that $f(z)$ can be written in the form $f(z) = z^2 g(z)$, where $g(z)$ is analytic at $z = 0$.

Notice: For problems 6-8, you need to give a closed form or explicitly write out at least the first three terms of the power series as your answer to each question. For example,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \cdots.$$

6. (7+8 pts) Find the Taylor expansions of the following functions with center z_0:

(a) $\frac{1}{1-z}$, $z_0 = i$, (b) $\mathrm{Log}(\frac{1-z}{1+z})$, $z_0 = 0$.

7. (15 pts) Expand $f(z) = \frac{z}{(z+1)^2(z-2)}$ in a Maclaurin series and give the radius of convergence.

8. (15 pts) Find the Laurent series for $f(z) = \frac{1}{(z-2)(z-1)^3}$ in the region of $0 < |z-2| < 1$.