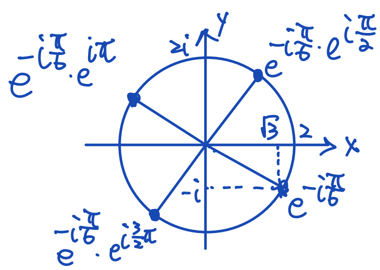
Complex Analysis 2021 Solution
Exam-1, D.C. Chang

#1. (a)

$$-8-8\sqrt{3}i = 16\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = 16\,e^{i(-\frac{2}{3}\pi+2n\pi)}\quad, n=0,\pm1,\pm2,\cdots$$

$$(-8-8\sqrt{3}i)^{\frac{1}{4}} = 2\cdot e^{i(-\frac{\pi}{6}+\frac{n\pi}{2})}\quad, n=0,1,2,3$$



$$= 2\times\begin{cases} e^{-i\frac{\pi}{6}} & = 2\cdot\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=\sqrt{3}-i \\ e^{-i\frac{\pi}{6}}\cdot e^{i\frac{\pi}{2}} & = (\sqrt{3}-i)\cdot i = 1+\sqrt{3}i \\ e^{-i\frac{\pi}{6}}\cdot e^{i\pi} & = (\sqrt{3}-i)\cdot(-1) = -\sqrt{3}+i \\ e^{-i\frac{\pi}{6}}\cdot e^{i\frac{3}{2}\pi} & = (\sqrt{3}-i)\cdot(-i) = -1-\sqrt{3}i \end{cases}$$

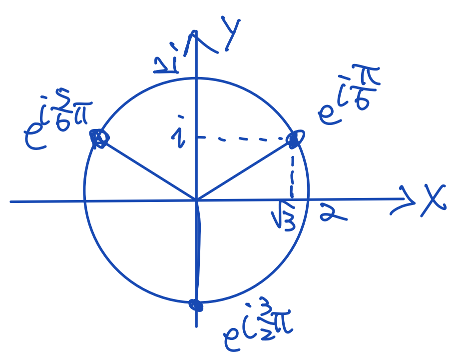
(b)

$$(2+2i)^{\frac{2}{3}} = \left(2^{\frac{3}{2}}\cdot\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)\right)^{\frac{2}{3}}$$

$$= 2\cdot\left(e^{i(\frac{\pi}{4}+2n\pi)}\right)^{\frac{2}{3}}\quad, n=0,1,2$$

$$= 2\cdot e^{i(\frac{\pi}{6}+\frac{4n\pi}{3})}$$

$$= 2\times\begin{cases} e^{i\frac{\pi}{6}} & = \sqrt{3}+i \\ e^{i\frac{\pi}{6}}\cdot e^{i\frac{4}{3}\pi} & = 2e^{i\frac{3}{2}\pi} = -2i \\ e^{i\frac{\pi}{6}}\cdot e^{i\frac{8}{3}\pi} & = 2e^{i\frac{5}{6}\pi} = -\sqrt{3}+i \end{cases}$$



#2.

$$(e^{i\theta})^4 = e^{i4\theta} = \cos 4\theta + i\sin 4\theta$$
$$= (\cos\theta + i\sin\theta)^4$$
$$= (\cos^2\theta - \sin^2\theta + 2i\cos\theta\cdot\sin\theta)^2$$
$$= (\cos^2\theta - \sin^2\theta)^2 - 4\cos^2\theta\cdot\sin^2\theta$$
$$+ 4i\cdot(\cos^2\theta - \sin^2\theta)\cdot\cos\theta\cdot\sin\theta$$

$$\cos 4\theta = (\cos^2\theta - \sin^2\theta)^2 - 4\cos^2\theta\sin^2\theta$$
$$= \cos^4\theta + \sin^4\theta - 2\cos^2\theta\sin^2\theta - 4\cos^2\theta\sin^2\theta$$
$$= \cos^4\theta + (1-\cos^2\theta)^2 - 6\cos^2\theta\cdot(1-\cos^2\theta)$$
$$= \cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta$$
$$= \underline{8\cos^4\theta - 8\cos^2\theta + 1} \quad ※$$

#3.

Let $f(z) = z^5 + 2z^4 + 4z^3 + 8z^2 + 16z + 32$

$\because f(z) = 0$

$\therefore (z-2)\cdot f(z) = z^6 - 64 = 0$.

$\Rightarrow$ the solutions of $f(z)=0$ include those of $z^6 - 64 = 0$, except $z = 2$.

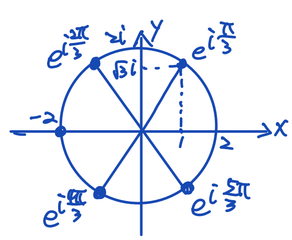
The solutions of $z^6 = 64$ are $z = 2\cdot e^{i\frac{n\pi}{3}}$, $n = 0, 1, 2, \cdots, 5$.

Then, the solutions of $f(z) = 0$ are

$2e^{i\frac{\pi}{3}}$, $2e^{i\frac{2\pi}{3}}$, $2e^{i\pi}$, $2e^{i\frac{4}{3}\pi}$, $2e^{i\frac{5}{3}\pi}$.

i.e.

$\underline{1+\sqrt{3}i,\ -1+\sqrt{3}i,\ -2,\ -1-\sqrt{3}i,\ 1-\sqrt{3}i}$ ※



#4.

$z = x+iy$, $x, y \in \mathbb{R}$

$f(z) = e^{i\bar{z}} = e^{i(x-iy)} = e^{y+ix}$

$u = e^y \cos x$, $v = e^y \sin x$

$u_x = -e^y \sin x$, $v_y = e^y \sin x$

$u_y = e^y \cos x$, $v_x = e^y \cos x$

If Cauchy-Riemann equations can be satisfied.

then $\begin{cases} u_x = v_y, & e^y \sin x = 0 \text{ ----- ①} \\ u_y = -v_x, & e^y \cos x = 0 \text{ ---- ②} \end{cases}$

From ① : $\sin x = 0$, $x = 2n\pi$, $n = 0, \pm 1, \pm 2, \cdots$

From ② : $\cos x = 0$, $x = n\pi$, $n = \pm 1, \pm 3, \pm 5, \cdots$

We conclude that the Cauchy-Riemann equations are not satisfied, and then f(z) is not analytic. #

#5.

If v is a harmonic conjugate for u,

then ① $u_x = v_y$ and ② $u_y = -v_x$.

Let w be a harmonic conjugate for v,

then, $v_x = w_y$ and $v_y = -w_x$.

$$\begin{aligned} \Rightarrow w &= \int v_x \, dy + p(x) \\ &= -\int u_y \, dy + p(x) \quad ② \\ &= -u + p(x) \end{aligned}$$

$$\begin{aligned} \because w_x &= -u_x + p'(x) \\ &= -v_y \\ &= -u_x \quad ① \end{aligned}$$

$\therefore p'(x) = 0$, $p(x) = C$ ... constant.

$\Rightarrow w = -u + C$

#6.

$$e^{z^2}=e^{(x+iy)^2}=e^{x^2-y^2+i\cdot 2xy}$$

$$=e^{x^2-y^2}\cdot\cos 2xy+i\cdot e^{x^2-y^2}\cdot\sin 2xy$$

$$u=\operatorname{Im}\{e^{z^2}\}=e^{x^2-y^2}\cdot\sin 2xy$$

(a) $u_x=2x\cdot e^{x^2-y^2}\sin 2xy+2ye^{x^2-y^2}\cos 2xy$

$$u_{xx}=2e^{x^2-y^2}\sin 2xy+4x^2e^{x^2-y^2}\sin 2xy$$
$$+4xye^{x^2-y^2}\cos 2xy+4xye^{x^2-y^2}\cos 2xy$$
$$-4y^2e^{x^2-y^2}\sin 2xy$$

$$u_y=-2ye^{x^2-y^2}\sin 2xy+2xe^{x^2-y^2}\cos 2xy$$

$$u_{yy}=-2e^{x^2-y^2}\sin 2xy+4y^2e^{x^2-y^2}\sin 2xy$$
$$-4xye^{x^2-y^2}\cos 2xy-4xye^{x^2-y^2}\cos 2xy$$
$$-4x^2e^{x^2-y^2}\sin 2xy$$

$$u_{xx}+u_{yy}=0$$

(b) $\because$ e^{z^2} is a entire function

u is a harmonic conjugate for $\operatorname{Re}\{e^{z^2}\}$

By problem #5,

$-\operatorname{Re}\{e^{z^2}\}+C=-e^{x^2-y^2}\cos 2xy+C$ is a harmonic conjugate for u, where C is a constant.

#7.

(a) $u = \alpha x^3 + \beta x^2 y + \gamma x y^2 + \delta y^3$

$u_x = 3\alpha x^2 + 2\beta x y + \gamma y^2$

$u_{xx} = 6\alpha x + 2\beta y$

$u_y = \beta x^2 + 2\gamma x y + 3\delta y^2$

$u_{yy} = 2\gamma x + 6\delta y$

$\because$ u is a harmonic function, then

$u_{xx} + u_{yy} = 0 \Rightarrow (6\alpha + 2\gamma)x + (2\beta + 6\delta)y = 0$

$\therefore \gamma = -3\alpha, \ \beta = -3\delta$

the most general harmonic polynomial

is $\underline{u = \alpha x^3 - 3\delta x^2 y - 3\alpha x y^2 + \delta y^3}$ #

(b) Let v be a harmonic conjugate for u

$u_x = v_y$ and $v_x = -u_y$

$\Rightarrow v_y = 3\alpha x^2 - 6\delta x y - 3\alpha y^2$

$\therefore v = 3\alpha x^2 y - 3\delta x y^2 - \alpha y^3 + p(x)$

$v_x = 6\alpha x y - 3\delta y^2 + p'(x)$

$= -u_y = 3\delta x^2 + 6\alpha x y - 3\delta y^2$

$\Rightarrow p'(x) = 3\delta x^2 \quad \therefore p(x) = \delta x^3 + c$, c: constant.

Then, $\underline{v = 3\alpha x^2 y - 3\delta x y^2 - \alpha y^3 + \delta x^3 + c}$ #