CO: Complex Analysis Exam-2, 2021/11/11

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#1.

$$\cos z = \frac{1}{2}(e^{iz}+e^{-iz}) = \frac{1}{2}(e^{-y+ix}+e^{y-ix})$$
$$= \frac{1}{2}[e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)]$$
$$= \cos x\cdot\frac{1}{2}(e^{y}+e^{-y}) + i\sin x\cdot\frac{1}{2}(e^{-y}-e^{y})$$
$$= \cos x\cdot\cosh y - i\sin x\cdot\sinh y$$

$$|\cos z|^2 = \cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y$$
$$= \cos^2 x(1+\sinh^2 y) + \sin^2 x\sinh^2 y$$
$$= \cos^2 x + \sinh^2 y$$

#2.

$$(1-i)^{2+i} = e^{(2+i)\cdot \ln(1-i)}$$
$$= e^{(2+i)\cdot[\ln\sqrt{2} + i(-\frac{\pi}{4}+2n\pi)]}, \quad n\in \mathbb{Z}.$$

principal value of $(1-i)^{2+i}$ is

$$\text{P.V.}\,(1-i)^{2+i} = e^{(2+i)[\ln\sqrt{2} - i\frac{\pi}{4}]}$$
$$= e^{2\ln\sqrt{2} + \frac{\pi}{4} + i\ln\sqrt{2} - i\frac{\pi}{2}}$$
$$= 2e^{\frac{\pi}{4}}\cdot e^{i\ln\sqrt{2}}\cdot(-i)$$
$$= 2e^{\frac{\pi}{4}}\left(\sin(\tfrac{1}{2}\ln 2) - i\cos(\tfrac{1}{2}\ln 2)\right)$$

#3.

$$f(z) \triangleq \operatorname{Ln} z = \operatorname{Ln}(re^{i\theta}) = \operatorname{Ln} r + i\theta$$

$$r > 0, \quad -\pi < \theta < \pi.$$

$$\therefore u(r,\theta) = \operatorname{Ln} r, \quad v(r,\theta) = \theta$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{\partial u}{\partial \theta} = 0$$

$$\frac{\partial v}{\partial r} = 0, \quad \frac{\partial v}{\partial \theta} = 1$$

the Cauchy-Riemann equations in polar coordinates satisfy that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Then

$$f'(z) = \frac{d}{dz}\operatorname{Ln} z = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$= e^{-i\theta} \cdot \frac{1}{r} = \frac{1}{re^{i\theta}}$$

$$= \frac{1}{z} \quad \#$$

#4.

$$z = Re^{i\theta}, \quad 0 \le \theta \le 2\pi, \quad dz = Ri \cdot e^{i\theta} d\theta$$

(a) $$\oint_C |z^m| dz = \int_0^{2\pi} |R^m e^{im\theta}| \cdot Ri e^{i\theta} d\theta$$

$$= i \int_0^{2\pi} R^{m+1} \cdot e^{i\theta} d\theta$$

$$= i R^{m+1} \cdot \frac{1}{i} e^{i\theta} \Big|_0^{2\pi}$$

$$= R^{m+1} (e^{i \cdot 2\pi} - 1) = \underline{0} \quad \#$$

(b) $$\oint_C |z^m| |dz| = \int_0^{2\pi} |R^m e^{im\theta}| |Ri e^{i\theta}| d\theta$$

$$= \int_0^{2\pi} R^{m+1} d\theta = \underline{2\pi R^{m+1}} \quad \#$$

(c) $$\oint_C \bar{z}^m dz = \int_0^{2\pi} R^m e^{-im\theta} \cdot i R e^{i\theta} d\theta$$

$$= i R^{m+1} \cdot \int_0^{2\pi} e^{i(1-m)\theta} d\theta$$

① $m = 1$, $$\oint_C \bar{z}^m dz = i \cdot R^2 \cdot \int_0^{2\pi} d\theta$$

$$= \underline{2\pi i \cdot R^2} \quad \#$$

② $m \ne 1$, $$\oint_C \bar{z}^m dz = i R^{m+1} \int_0^{2\pi} e^{i(1-m)\theta} d\theta$$

$$= \frac{R^{m+1}}{1-m} \left(e^{i(1-m)2\pi} - 1\right)$$

$$= \underline{0} \quad \#$$

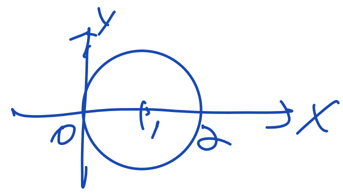
#5.

$$\left| \oint_{|z-1|=1} \frac{e^{z^2}}{z+1} dz \right| \leq \oint_{|z-1|=1} \left| \frac{e^{z^2}}{z+1} \right| |dz|$$

$$z = 1 + e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

$$|e^{z^2}| = |e^{(x+iy)^2}| = |e^{x^2-y^2+i2xy}| = e^{x^2-y^2}$$

$$\begin{cases} x = 1+\cos\theta \\ y = \sin\theta \end{cases}$$



$$x^2 - y^2 = 1 + 2\cos\theta + \cos^2\theta - \sin^2\theta$$
$$= 1 + 2\cos\theta + \cos^2\theta - 1 + \cos^2\theta$$
$$= 2\cos^2\theta + 2\cos\theta \leq 4$$

$$\therefore |e^{z^2}| \leq e^4$$

$$|z+1| \geq 1 - |z| = 1 - \sqrt{(1+\cos\theta)^2 + \sin^2\theta}$$
$$= 1 - \sqrt{2(1+\cos\theta)} \geq 1$$

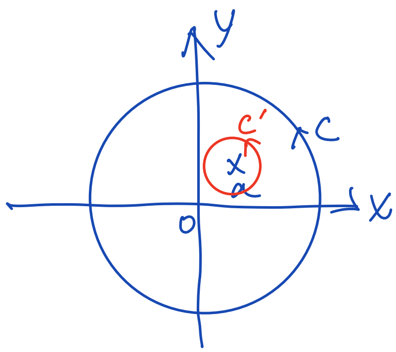
$$\therefore \frac{1}{|z+1|} \leq 1$$

$$\Rightarrow \left| \frac{e^{z^2}}{z+1} \right| \leq M = e^4$$

$$\therefore \left| \oint_{|z-1|=1} \frac{e^{z^2}}{z+1} dz \right| \leq M \cdot \oint_{|z-1|=1} |dz|$$
$$= 2\pi e^4 \ \#$$

#6

(a) With contour deformation,



$$\oint_C \frac{dz}{z-a} = \oint_{C'} \frac{dz}{z-a}$$

$C: |z|=1$, $C': |z-a|=R$

C' is within C.

on C', $z = a + Re^{i\theta}$

$0 \le \theta \le 2\pi$, $dz = Ri \cdot e^{i\theta} d\theta$

$$\oint_{C'} \frac{dz}{z-a} = \int_0^{2\pi} R^{-1} e^{-i\theta} \cdot Ri \cdot e^{i\theta} d\theta$$

$$= i \cdot \int_0^{2\pi} d\theta = 2\pi i$$

$$\therefore \oint_C \frac{dz}{z-a} = 2\pi i \ \#$$

(b) For $C: |z|=1$, $z = e^{i\theta}$, $0 \le \theta \le 2\pi$

$$\oint_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{i \cdot e^{i\theta}}{e^{i\theta} - a} d\theta$$

$$= \int_0^{2\pi} \frac{i e^{i\theta} \cdot (e^{-i\theta} - a)}{(e^{-i\theta} - a)(e^{i\theta} - a)} d\theta$$

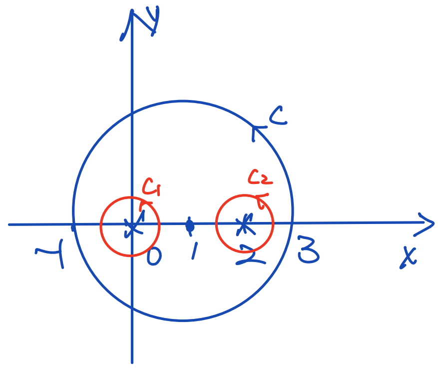
$$= \int_0^{2\pi} \frac{i \cdot (1 - a e^{i\theta})}{1 - 2a\cos\theta + a^2} d\theta$$

$$= \int_0^{2\pi} \frac{a\sin\theta}{1 - 2a\cos\theta + a^2} d\theta + i \cdot \int_0^{2\pi} \frac{1 - a\cos\theta}{1 - 2a\cos\theta + a^2} d\theta$$

$$= 0 + 2\pi i$$

$$\Longrightarrow \int_0^{2\pi} \frac{1 - a\cos\theta}{1 - 2a\cos\theta + a^2} d\theta = 2\pi \ \#$$

#7.



$$f(x) = \frac{1}{z^2(z^2-4)e^z}$$

$$\oint_C f(z)dz$$

$$= \oint_{C_1} \frac{1/(z^2-4)e^z}{z^2} dz + \oint_{C_2} \frac{1/z^2(z+2)e^z}{z-2} dz$$

$$\oint_{C_1} \frac{1/(z^2-4)e^z}{z^2} dz = \frac{2\pi i}{1!} \cdot \frac{d}{dz}\left(\frac{1}{(z^2-4)e^z}\right)\Bigg|_{z=0}$$

$$= 2\pi i \cdot \frac{-e^{-z}(z^2-4) - e^{-z} \cdot 2z}{(z^2-4)^2}\Bigg|_{z=0}$$

$$= 2\pi i \cdot \frac{4}{4^2} = \frac{\pi i}{2}$$

$$\oint_{C_2} \frac{1/z^2(z+2) \cdot e^z}{z-2} dz = 2\pi i \cdot \frac{1}{z^2(z+2)e^z}\Bigg|_{z=2}$$

$$= 2\pi i \cdot \frac{1}{16e^2} = \frac{\pi i}{8e^2}$$

$$\therefore \oint_C f(z)dz = \underline{\left(\frac{1}{2} + \frac{1}{8e^2}\right) \cdot \pi i} \ \#$$