Complex Analysis 2021

Exam-3 Solution P.C.Chang

#1.

(a) $z_n = \frac{n^2}{4^n}$

$$L = \lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^2}{4^{n+1}}}{\frac{n^2}{4^n}}\right| = \lim_{n\to\infty}\frac{n^2+2n+1}{4n^2}$$

$= \frac{1}{4} < 1$, absolutely converges !!

$\Rightarrow \sum_{n=0}^{\infty} z_n$ converges.

(b)

$$S_n = \sum_{k=1}^{n}(z_{k+1} - z_k)$$
$$= z_{n+1} - z_1$$

If $\{z_n\}_1^{\infty}$ converges, $\lim_{n\to\infty} z_n = C$.

$\because$ A convergent series is one whose sequence of partial sum converges.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} z_{n+1} - z_1$$
$= C - z_1 \implies$ converges !!

$\therefore$ $\{z_n\}_1^{\infty}$ converges $\Leftrightarrow \lim_{n\to\infty} S_n$ converges

i.e. $\{z_n\}_1^{n}$ converges $\Leftrightarrow \sum_{n=1}^{\infty}(z_{n+1} - z_n)$ converges.

#2.

(a) $z_n = \dfrac{(z-2-i)^{2n}}{2^{3n}}$

By root test, a convergent series

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = \left| \frac{(z-2-i)^2}{2^3} \right| = \frac{|z-2-i|^2}{8} < 1$$

$$\Rightarrow |z-(2+i)| < \sqrt{8} = 2\sqrt{2}$$

$\begin{cases} \text{center of the circle} : 2+i \\ \text{radius of convergence} : 2\sqrt{2} \end{cases}$

(b) $z_n = \dfrac{1}{n}\left(\dfrac{i}{1+i}\right)(z-i)^n$

By ratio test, a convergent series

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{n+1}\left(\frac{i}{1+i}\right)\cdot(z-i)^{n+1}}{\frac{1}{n}\left(\frac{i}{1+i}\right)(z-i)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n}{n+1}(z-i) \right| < 1$$

$$\Rightarrow |z-i| < 1$$

$\begin{cases} \text{center of the circle} : i \\ \text{radius of convergence} : 1 \end{cases}$

#3.

$$e^{z/2} = 1 + \frac{z}{2} + \frac{1}{2!}\left(\frac{z}{2}\right)^2 + \frac{1}{3!}\left(\frac{z}{2}\right)^3 + \frac{1}{4!}\left(\frac{z}{2}\right)^4 + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$e^{z/2}\cdot \sin z = \cdots + \frac{z}{2}\cdot\left(-\frac{z^3}{3!}\right) + \frac{1}{3!}\left(\frac{z}{2}\right)^3\cdot z + \cdots$$

$$= \cdots + \left(\frac{1}{48} - \frac{1}{12}\right) z^4 + \cdots$$

$$= \cdots - \frac{1}{16} z^4 + \cdots$$

$$\oint_{C:|z|=1} \frac{e^{z/2}\cdot \sin z}{z^5}\, dz = \oint_{C:|z|=1} \left(-\frac{1}{16}\right)\cdot \frac{dz}{z}$$

$$= -\frac{1}{16}\cdot 2\pi i = -\frac{\pi i}{8}$$ ※

#4.

(a)

$$f(z) = (z-1)e^{-3z}$$

$$= (z-1)e^{-3(z-1)-3}$$

$$= e^{-3}\cdot (z-1)e^{-3(z-1)}$$

$$= e^{-3}(z-1)\cdot \sum_{n=0}^{\infty} \frac{(-3(z-1))^n}{n!}$$

$$= e^{-3}\cdot \sum_{n=0}^{\infty} \frac{(-3)^n}{n!}\cdot (z-1)^{n+1}$$

for $|z| < \infty$

(b) $f(z) = \frac{1}{z} \cdot \frac{z+1}{(z-2)^2} = \frac{1}{z} \cdot \left(\frac{1}{z-2} + \frac{3}{(z-2)^2} \right)$

For a Taylor series,

(i) $\frac{1}{z-2} = -\frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$

$\left|\frac{z}{2}\right| < 1$

(ii) $\frac{3}{(z-2)^2} = -3 \cdot \frac{d}{dz} \left(\frac{1}{z-2} \right)$

$= \frac{3}{2} \cdot \frac{d}{dz} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \qquad \left|\frac{z}{2}\right| < 1$

$= \frac{3}{2} \cdot \sum_{n=0}^{\infty} \frac{1}{2}(n+1) \cdot \left(\frac{z}{2}\right)^n$

$\therefore f(z) = \frac{1}{z} \left[-\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n + \frac{3}{2} \sum_{n=0}^{\infty} \frac{n+1}{2} \cdot \left(\frac{z}{2}\right)^n \right]$

$= -\frac{1}{4} \sum_{n=0}^{\infty} \left(1 - \frac{3}{2}(n+1)\right) \cdot \left(\frac{z}{2}\right)^{n-1} \quad , \; 0 < |z| < 2$

#5.

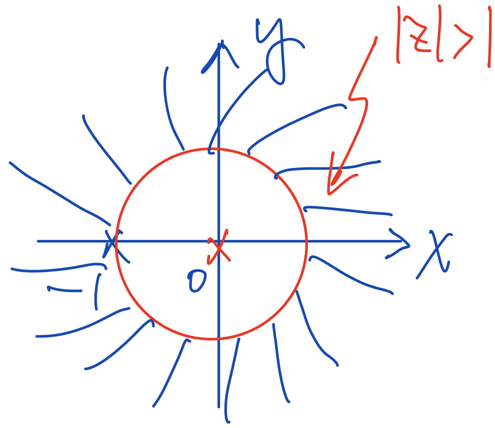
$$f(z) = \frac{1}{z+z^2} = \frac{1}{z}\cdot\frac{1}{1+z}, \quad z \neq 0$$

(a) $1 < |z|$,

$$f(z) = \frac{1}{z}\cdot\frac{1}{z}\cdot\frac{1}{1+\frac{1}{z}} \qquad \left|\frac{1}{z}\right| < 1$$

$$= \frac{1}{z^2}\cdot\sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^{n+2}$$



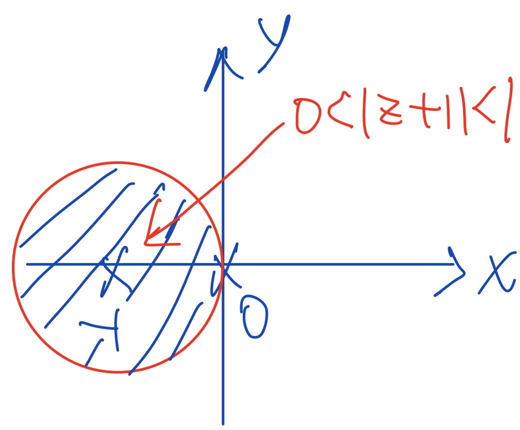
(b) $0 < |z+1| < 1$

$$f(z) = \frac{1}{z+1}\cdot\frac{1}{z+1-1}$$

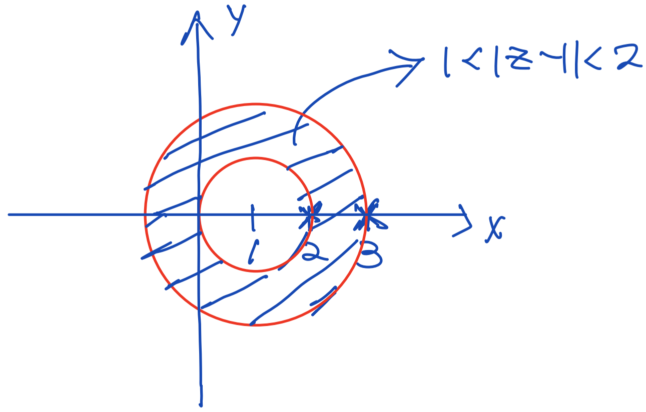
$$= \frac{-1}{z+1}\cdot\frac{1}{1-(z+1)}$$

$$= \frac{-1}{z+1}\cdot\sum_{n=0}^{\infty}(z+1)^n \qquad |z+1|<1$$

$$= -\sum_{n=0}^{\infty}(z+1)^{n-1}$$



#6. $f(z) = \dfrac{1}{(z-2)(z-3)} = \dfrac{-1}{z-2} + \dfrac{1}{z-3}$



(i) $\dfrac{-1}{z-2} = -\dfrac{1}{(z-1)-1} = \dfrac{-1}{z-1} \cdot \dfrac{1}{1-\frac{1}{z-1}}$

$= \dfrac{-1}{z-1} \cdot \displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{z-1}\right)^n, \quad \left|\frac{1}{z-1}\right| < 1$

$= -\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{z-1}\right)^{n+1}$

(ii) $\dfrac{1}{z-3} = \dfrac{1}{(z-1)-2} = -\dfrac{1}{2} \cdot \dfrac{1}{1-\frac{z-1}{2}}$

$= -\dfrac{1}{2} \cdot \displaystyle\sum_{n=0}^{\infty} \left(\frac{z-1}{2}\right)^n, \quad \left|\frac{z-1}{2}\right| < 1$

$\therefore f(z) = -\displaystyle\sum_{n=0}^{\infty} \left[\left(\frac{1}{z-1}\right)^{n+1} + \frac{1}{2}\left(\frac{z-1}{2}\right)^n \right]$