Complex Analysis Solution.

Exam-4. 2022/1/13 D. C. Chang

#1. $I=\int_0^{2\pi} \frac{\cos\theta}{2+\sin\theta}\,d\theta$

Let $C: z=e^{i\theta},\ 0\le\theta\le 2\pi,\ d\theta=\frac{1}{iz}dz$

$\cos\theta=\frac{1}{2}(z+z^{-1}),\quad \sin\theta=\frac{1}{2i}(z-z^{-1})$

$$I=\oint_C \frac{\frac{1}{2}(z+z^{-1})}{2+\frac{1}{2i}(z-z^{-1})}\cdot\frac{2i\cdot z}{2i\cdot z}\cdot\frac{dz}{iz}$$

$$=\oint_C \frac{z^2+1}{z\cdot(z^2+4iz-1)}dz=\oint_C f(z)dz$$

$$z^2+4iz-1=[z+(2+\sqrt{3})i][z+(2-\sqrt{3})i]$$

$z=0$ and $z=-(2-\sqrt{3})i$ are inside C.

$$\operatorname*{Res}_{z=0} f(z)=\left.\frac{z^2+1}{z^2+4iz-1}\right|_{z=0}=-1$$

$$\operatorname*{Res}_{z=-(2-\sqrt{3})i} f(z)=\left.\frac{z^2+1}{z\cdot(z+(2+\sqrt{3})i)}\right|_{z=-(2-\sqrt{3})i}$$

$$=\frac{-(3-2\sqrt{3})}{-3+2\sqrt{3}}=1$$

$$I=2\pi i\cdot\left(\operatorname*{Res}_{z=0} f(z)+\operatorname*{Res}_{z=-(2-\sqrt{3})i} f(z)\right)=0.$$

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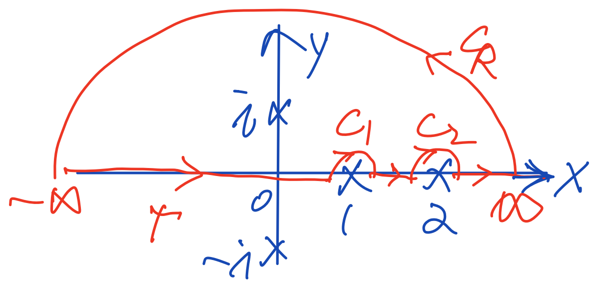
#2. $I = \text{p.v.}\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+1)\cdot(x^2-3x+2)}\,dx$

Let $f(z) = \frac{ze^{iz}}{(z^2+1)(z^2-3z+2)}$

$= \frac{z\cos z}{(z^2+1)(z^2-3z+2)} + i\cdot\frac{z\sin z}{(z^2+1)(z^2-3z+2)}$

$z=i$ is in upper half-plane, $z=1,2$ are on x-axis



$C = C_R + \Gamma + C_1 + C_2$

$\oint_C = \int_\Gamma + \int_{C_1} + \int_{C_2}$

$\therefore \int_\Gamma = \oint_C - \int_{C_1} - \int_{C_2}$

$-\int_{C_1} = \pi i\cdot \operatorname*{Res}_{z=1} f(z) = \pi i\cdot\frac{ze^{iz}}{(z^2+1)(z-2)}\Big|_{z=1} = -\frac{\pi i e^{i}}{2}$

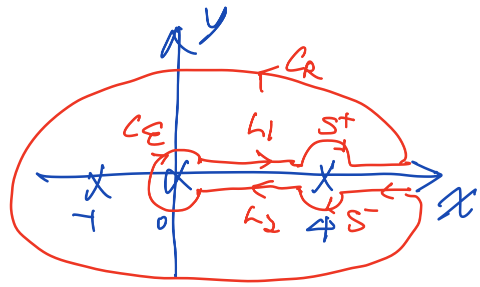
$-\int_{C_2} = \pi i \operatorname*{Res}_{z=2} f(z) = \pi i\cdot\frac{ze^{iz}}{(z^2+1)(z-1)}\Big|_{z=2} = \pi i\cdot\frac{2e^{i2}}{5}$

$\oint_C = 2\pi i\cdot \operatorname*{Res}_{z=i} f(z) = 2\pi i\,\frac{ze^{iz}}{(z+i)(z^2-3z+2)}\Big|_{z=i} = \frac{\pi}{e}\cdot\frac{-3+i}{10}$

$I = \text{Im}\{\int_\Gamma\} = \underline{\frac{\pi}{10e} + \frac{2\pi}{5}\cos 2 - \frac{\pi}{2}\cos 1}$ ※

#3 $I = P.V. \int_0^\infty \frac{dx}{\sqrt{x}(x^2-3x-4)}$

Let $f(z) = \frac{z^{-1/2}}{(z+1)(z-4)}$



$C = C_R + C_\varepsilon + L_1 + L_2 + S^+ + S^-$

$\oint_C = \int_{C_R + C_\varepsilon + L_1 + L_2 + S^+ + S^-}$

Note: $\int_{C_R} = \int_{C_\varepsilon} = 0$

$\int_{L_1} = \int_0^\infty f(x)\,dx$

For L_2, $z = x e^{i2\pi}$, $dz = dx$

$\int_{L_2} = \int_\infty^0 \frac{x^{-1/2} \cdot e^{-i\pi}}{(x+1)(x-4)}\,dx = -e^{-i\pi} \cdot \int_{L_1}$

$\int_{S^+} = -\pi i \cdot \operatorname{Res}_{z=4} f(z) = -\pi i \cdot \left.\frac{z^{-1/2}}{z+1}\right|_{z=4} = \frac{-\pi i}{10}$

$\int_{S^-} = \int_{S^+} f(ze^{i2\pi})\,dz = e^{-i\pi} \cdot \int_{S^+} = e^{-i\pi}\left(\frac{-\pi i}{10}\right)$

$\oint_C = 2\pi i \cdot \operatorname{Res}_{z=-1} f(z) = 2\pi i \cdot \left.\frac{z^{-\frac{1}{2}}}{z-4}\right|_{z=-1} = 2\pi i \cdot \frac{1}{-5i} = -\frac{2\pi}{5}$

$\because e^{-i\pi} = -1$

$\therefore 2 \cdot \int_{L_1} = -\frac{2\pi}{5} \Rightarrow \int_0^\infty f(x)\,dx = -\frac{\pi}{5}$ #

#4 $s^4-2s^3+2s^2-2s+1 = s^2(s^2-2s+1)+s^2-2s+1$

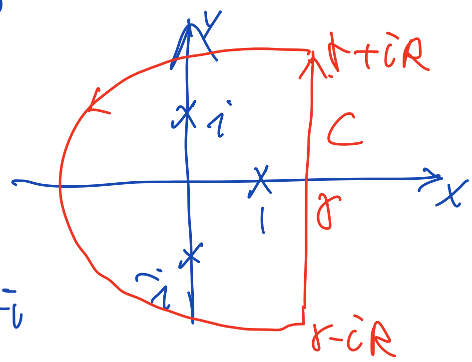
$$=(s^2+1)(s-1)^2$$

$$F(s)=\frac{1}{(s^2+1)(s-1)^2}$$

$$\mathcal{L}^{-1}\{F(s)\}=\frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR} e^{st}F(s)\,ds \quad ,\gamma>1,\ R\to\infty$$

$$=\frac{1}{2\pi i}\oint_C e^{st}F(s)\,ds$$

$s=-i,+i,1$ are inside C



$$\operatorname*{Res}_{s=-i} e^{st}F(s)=\left.\frac{e^{st}}{(s-i)(s-1)^2}\right|_{s=-i}$$

$$=\frac{e^{-it}}{4}$$

$$\operatorname*{Res}_{s=i} e^{st}F(s)=\left.\frac{e^{st}}{(s+i)(s-1)^2}\right|_{s=i}=\frac{e^{it}}{4}$$

$$\operatorname*{Res}_{s=1} e^{st}F(s)=\frac{d}{dz}\left.\frac{e^{st}}{s^2+1}\right|_{s=1}=\left.\frac{te^{st}(s^2+1)-2se^{st}}{(s^2+1)^2}\right|_{s=1}$$

$$=\frac{1}{2}te^t-\frac{1}{2}e^t$$

$$f(t)=\frac{1}{2}te^t-\frac{1}{2}e^t+\frac{1}{4}(e^{it}+e^{-it})$$

$$=\frac{1}{2}te^t-\frac{1}{2}e^t+\frac{1}{2}\cos t$$

※

#5. $f(t) = te^{-2t}, \quad t > 0$

(a) Fourier transform of $f(t)$

$$F(w) = \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt$$
$$= \frac{1}{2\pi}\int_0^{\infty} te^{-2t}\cdot e^{-iwt}\, dt$$
$$= \frac{1}{2\pi}\int_0^{\infty} t\cdot e^{-(2+iw)t}\, dt$$

From integration-by-parts,

$$\int u\,dv = uv - \int v\cdot du$$
$$u = t, \quad v = \frac{e^{-(2+iw)t}}{-(2+iw)}$$

We have that

$$F(w) = \frac{1}{2\pi}\left(\frac{t\, e^{-(2+iw)t}}{-(2+iw)}\Big|_0^{\infty} - \frac{e^{-(2+iw)t}}{(2+iw)^2}\Big|_0^{\infty} \right)$$
$$= \frac{1}{2\pi}\cdot\frac{1}{-(2+iw)}\cdot \underbrace{\frac{t}{e^{(2+iw)t}}\Big|_{t\to\infty}}_{\to 0 \text{ by L'Hopital's rule}} + \frac{1}{2\pi}\cdot\frac{1}{(2+jw)^2}$$
$$= \frac{1}{2\pi(2+iw)^2} \; \#$$

Note that: $\lim_{t\to\infty} \left|e^{-(2+iw)t}\right| = \lim_{t\to\infty}\left|e^{-2t}\right|\cdot\left|e^{-iwt}\right| = 0$

(b) $F(w) = \dfrac{1}{2\pi(2+iw)^2}$

$$f(t) = \int_{-\infty}^{\infty} F(w)\cdot e^{iwt}\, dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iwt}}{(2+iw)^2}\, dw$$

Let $g(z) = \dfrac{e^{izt}}{(2+iz)^2} = \dfrac{-e^{izt}}{(z-2i)^2}$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\,dx$$

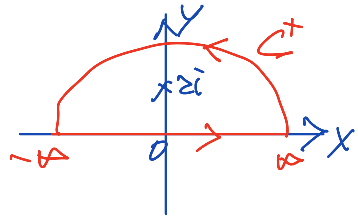
① $t > 0$.

$$\int_{-\infty}^{\infty} g(x)\,dx = \oint_{C^+} g(z)\,dz$$

$$= 2\pi i \cdot \operatorname*{Res}_{z=2i} g(z)$$

$$= 2\pi i \cdot \lim_{z\to 2i} \frac{d}{dz}\left(-e^{izt}\right)$$

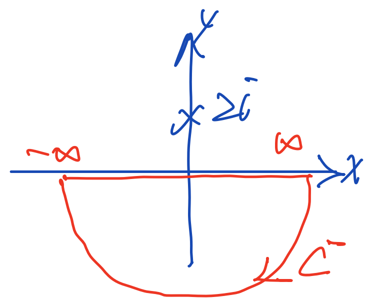
$$= -2\pi i \cdot it\cdot e^{izt}\Big|_{z=2i} = 2\pi t\cdot e^{-2t}$$



② $t<0$

$$\int_{-\infty}^{\infty} g(x)\,dx = \oint_{C^-} g(z)\,dz$$

$$= 0$$



$$\Rightarrow f(t) = \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty} g(x)\,dx$$

$$= te^{-2t},\quad t>0$$

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