CO2013: Complex Analysis, Exam-2, Fall 2021

Department of Communication Engineering, National Central University
Prof. Dah-Chung Chang (office: E1-311, e-mail: dcchang@ce.ncu.edu.tw)

Notice:
a) Term grading policy: Exam-2×25%.
b) Total 100 points in this exam.
c) Exam Time: 1:00PM−2:50PM, Nov. 11, 2021.

1. (10 pts) Let $z = x + iy$, where $x, y \in \mathbb{R}$. Define that
$$\sinh y = \frac{e^y - e^{-y}}{2} \qquad \text{and} \qquad \cosh y = \frac{e^y + e^{-y}}{2}.$$
Prove that $|\cos z|^2 = \cos^2 x + \sinh^2 y$.

2. (10 pts) Express the principal value of the complex number $(1-i)^{2+i}$ in the format of $x + iy$, where $x, y \in \mathbb{R}$.

3. (15 pts) Use the Cauchy-Riemann equations in polar coordinates to show that the derivative of $\operatorname{Ln} z$ is $1/z$, where $\operatorname{Ln} z$ denotes the principal branch of the complex logarithm $\log z$.

4. (15 pts) Let C be the circle $\{z : |z| = R\}$ with counterclockwise orientation. Evaluate the following integrals for $m = 0, \pm 1, \pm 2, \cdots$:
(a) $\oint_C |z^m| dz$, (b) $\oint_C |z^m||dz|$, (c) $\oint_C \bar{z}^m dz$.

5. (15 pts) Show that
$$\left| \oint_{|z-1|=1} \frac{e^{z^2}}{z+1} dz \right| \leq 2\pi e^4.$$

6. (20 pts) Let a be a real number and $|a| < 1$.
(a) Show that $\displaystyle\oint_{C:|z|=1} \frac{dz}{z-a} = 2\pi i$,
(b) From (a), show that $\displaystyle\int_0^{2\pi} \frac{1 - a\cos\theta}{1 - 2a\cos\theta + a^2} d\theta = 2\pi$.

7. (15 pts) Evaluate $\displaystyle\oint_{C:|z-1|=2} \frac{dz}{z^2(z^2-4)e^z}$.