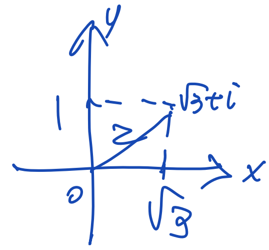
Complex Analysis 2022
Exam-1 Date 2022/10/18
D. C. Chang

#1.

(a) $\log(\sqrt{3}+i) = \log\left(2\cdot e^{i(\frac{\pi}{6}+2k\pi)}\right)$

$= \mathrm{Log}\,2 + i\left(\frac{\pi}{6}+2k\pi\right)$

$k=0,\pm1,\pm2,\dots$



(b)

$(1+i)^{3+4i} = e^{(3+4i)\ln(1+i)}$

$= e^{(3+4i)\cdot\left(\log\sqrt{2}+i\cdot\left(\frac{\pi}{4}+2k\pi\right)\right)}$

$= e^{3\log\sqrt{2}-\pi-8k\pi+i\left(4\log\sqrt{2}+3\pi/4+6k\pi\right)}$

$= e^{3\log\sqrt{2}-\pi-8k\pi}\cdot\left[\cos\left(4\log\sqrt{2}+\frac{3\pi}{4}\right)\right.$

$\left.+ i\sin\left(4\log\sqrt{2}+\frac{3\pi}{4}\right)\right]$, $k=0,\pm1,\pm2,\dots$

#2.

$u+iv$ is analytic, then

$u_x = v_y$, $u_y = -v_x$

If $u-iv$ is also analytic, then

$u_x = -v_y$, $u_y = v_x$

$\Rightarrow u_x=0, v_y=0, u_y=0, v_x=0$

$\Rightarrow$ u is a constant
and v is a constant

$g(z) = a+ib$, $a, b\in\mathbb{R}$, complex number

#3. Let $u = e^x \cos y + e^y \cos x + xy$

v is the harmonic conjugate of u

then $u_x = v_y$, $u_y = -v_x$

$u_x = e^x \cos y - e^y \sin x + y = v_y$

$\therefore v = e^x \sin y - e^y \sin x + \frac{1}{2} y^2 + c(x)$

$\because v_x = e^x \sin y - e^y \sin x + c'(x)$

$= -u_y = e^x \sin y - e^y \cos x - x$

$\Rightarrow c'(x) = -x \quad \therefore c(x) = -\frac{1}{2} x^2 + d$, d: constant

We have, $v = e^x \sin y - e^y \sin x + \frac{1}{2} y^2 - \frac{1}{2} x^2 + d$

#4.

$p = |z_1|$, $q = |z_2|$

$m = |z_1 + z_2|$, $n = |z_1 - z_2|$

$$
\begin{aligned}
m^2 + n^2 &= |z_1 + z_2|^2 + |z_1 - z_2|^2 \\
&= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\
&= |z_1|^2 + |z_2|^2 + \cancel{z_1 \bar{z}_2} + \cancel{z_2 \bar{z}_1} + |z_1|^2 + |z_2|^2 \\
&\quad - \cancel{z_1 \bar{z}_2} - \cancel{z_2 \bar{z}_1} \\
&= 2|z_1|^2 + 2|z_2|^2 \\
&= 2(p^2 + q^2)
\end{aligned}
$$

#5 Let $p = \operatorname{cis}\frac{2\pi}{n}$

With geometric series,

$$1+p+p^2+\cdots+p^{n-1} = \frac{1-p^{n+1}}{1-p}$$

$$1+\operatorname{cis}\frac{2\pi}{n}+\operatorname{cis}\frac{4\pi}{n}+\cdots+\operatorname{cis}\frac{2\pi(n-1)}{n}$$

$$= \frac{1-\operatorname{cis}\left(\frac{2\pi\cdot\cancel{n}}{\cancel{n}}\right)}{1-\operatorname{cis}\frac{2\pi}{n}} = 0$$

$$\therefore \begin{cases} \cos\frac{2\pi}{n}+\cos\frac{4\pi}{n}+\cdots+\cos\frac{2\pi(n-1)}{n} = -1 \\ \sin\frac{2\pi}{n}+\sin\frac{4\pi}{n}+\cdots+\sin\frac{2\pi(n-1)}{n} = 0 \end{cases}$$

#6

$$\cosh z = \frac{e^{z}+e^{-z}}{2} = \frac{e^{x}\operatorname{cis}y+e^{-x}\operatorname{cis}(-y)}{2}$$

$$= \frac{e^{x}+e^{-x}}{2}\cos y + i\cdot\frac{e^{x}-e^{-x}}{2}\sin y$$

$$= \cosh x\cos y + i\sinh x\sin y$$

$$|\cosh z|^2 = \cosh^2 x\cos^2 y+\sinh^2 x\sin^2 y$$

$$= \cosh^2 x(1-\sin^2 y)+(\cosh^2 x-1)\cdot\sin^2 y$$

$$= \cosh^2 x-\sin^2 y$$

7

(a) $z = \cosh w = \dfrac{e^w + e^{-w}}{2} = \dfrac{p + p^{-1}}{2}$, $p = e^w$

$p^2 - 2zp + 1 = 0$

$p = z + (z^2 - 1)^{1/2} = e^w$

$\therefore w = \cosh^{-1} z = \log(z + (z^2-1)^{1/2})$

(b) $\cosh^2 z = -1$, $\cosh z = \pm i$

$z = \cosh^{-1}(\pm i)$

① $z = \cosh^{-1}(i)$

$= \log(i \pm i\sqrt{2})$

$= \begin{cases} \log(\sqrt{2}+1) + i\left(\frac{\pi}{2} + 2k\pi\right) \\ \log(\sqrt{2}-1) + i\left(-\frac{\pi}{2} + 2k\pi\right) \end{cases}$

② $z = \cosh^{-1}(-i)$

$= \log(-i \pm i\sqrt{2})$

$= \begin{cases} \log(\sqrt{2}-1) + i\left(\frac{\pi}{2} + 2k\pi\right) \\ \log(\sqrt{2}+1) + i\left(-\frac{\pi}{2} + 2k\pi\right) \end{cases}$

$k = 0, \pm 1, \pm 2, \cdots$