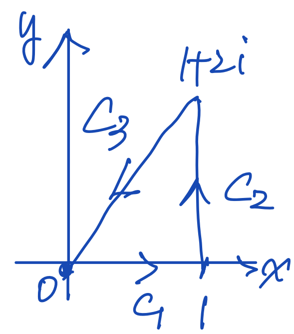
Complex Analysis 2022 Exam-2

D. C. Chang Dec. 8, 2022

#1.

$$\oint_C (z^2 + e^z + Re(z))\, dz$$

$$= \oint_C z^2 dz + \oint_C e^z dz + \oint_C Re(z)\, dz$$



$$\oint_C z^2 dz = \oint_C e^z dz = 0$$

$$\oint_C Re(z)\, dz = \oint_{C_1+C_2+C_3} Re(z)\, dz \qquad C = C_1 + C_2 + C_3$$

(a) $C_1 : \begin{cases} x = t \\ y = 0 \end{cases}, \quad 0 \le t \le 1 \qquad dz = dt$

$$\oint_{C_1} Re(z)\, dz = \int_0^1 t \cdot dt = \frac{1}{2}$$

(b) $C_2 : \begin{cases} x = 1 \\ y = 2i \cdot t \end{cases} \quad 0 \le t \le 1, \quad dz = 2i \cdot dt$

$$\oint_{C_2} Re(z)\, dz = \int_0^1 1 \cdot 2i \cdot dt = 2i$$

(c) $C_3 : \begin{cases} x = 1 - t \\ y = (2-2t)i \end{cases}, \quad 0 \le t \le 1, \quad dz = (-1-2i)dt$

$$\oint_{C_3} Re(z)\, dz = \int_0^1 (1-t)(-1-2i)\, dt \qquad u = 1-t$$

$$= (1+2i) \cdot \int_1^0 u\, du = -\frac{1+2i}{2}$$

$$\therefore \oint_C Re(z)\, dz = \frac{1}{2} + 2i - \frac{1}{2} - i = i \;\#$$

#2. (a) $f(z) = \dfrac{e^{-z} \cdot \sin z}{z^3}$ has a pole of order 3 at $z=0$.

$$\oint_C f(z)\,dz = \frac{2\pi i}{2!} \cdot \left(e^{-z} \sin z\right)'' \Big|_{z=0}$$

$$\left(e^{-z} \sin z\right)'' \Big|_{z=0} = -2e^{-z} \cos z \Big|_{z=0} = -2$$

$$\therefore \oint_C f(z)\,dz = \frac{2\pi i}{2} \cdot (-2) = -2\pi i.$$

(b) $z^2 + 3z - 4 = (z-1)(z+4)$

$f(z) = \dfrac{z^2+3z-2i}{z^2+3z-4}$ has simple poles $z = 1, -4$

(i) $$\oint_{C:|z|=2} f(z)\,dz = \oint_C \frac{\frac{z^2+3z-2i}{z+4}}{z-1}\,dz$$

$$= 2\pi i \left(\frac{z^2+3z-2i}{z+4}\right)_{z=1} = 2\pi i \cdot \frac{4+2i}{5} = \frac{8\pi i - 4\pi}{5}$$

(ii) $$\oint_{C:|z+5|=3} f(z)\,dz = \oint_C \frac{\frac{z^2+3z-2i}{z-1}}{z+4}\,dz$$

$$= 2\pi i \cdot \left(\frac{z^2+3z-2i}{z-1}\right)_{z=-4}$$

$$= 2\pi i \cdot \frac{4+2i}{-5} = \frac{4\pi - 8\pi i}{5}$$

#3 (a) $z_k = \frac{4^k}{3k+7}(z-2+3i)^{2k}$

By root test,

$$L = \lim_{k\to\infty}\sqrt[k]{|z_k|} = \lim_{k\to\infty}\sqrt[k]{\left|\frac{4^k}{3k+7}\cdot(z-2+3i)^{2k}\right|}$$

$$= 4\cdot|z-2+3i|^2\cdot\lim_{k\to\infty}\frac{1}{\sqrt[k]{3}}\cdot\frac{1}{\sqrt[k]{k}}$$

Note: $\lim_{k\to\infty}\frac{1}{\sqrt[k]{3}} = 1$

$$\lim_{k\to\infty}\frac{1}{k^{1/k}} = \frac{1}{\lim_{k\to\infty}k^{1/k}} = \frac{1}{\lim_{k\to\infty}e^{\ln k/k}}$$

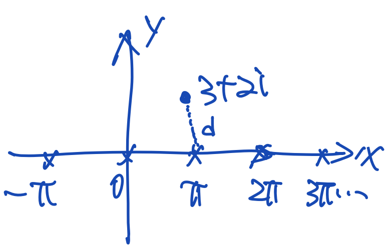
$$= \frac{1}{e^{\lim_{k\to\infty}\frac{1}{k}}} = 1$$

For convergence, $L<1$

$\therefore\ 4\cdot|z-2+3i|^2<1 \Rightarrow \underbrace{|z-(2-3i)|<\frac{1}{2}}$ ※

(b) $f(z) = \csc z = \frac{1}{\sin z}$

$\sin z = 0$ for $z = n\pi$, $n = 0, \pm1, \pm2, \cdots$

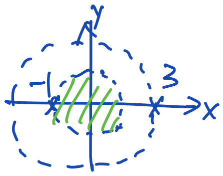


circle of convergence:

$|z-(3+2i)| < d$

$d = |3+2i-\pi| = \sqrt{4+(0.1416)^2} \doteq 2.005$

#4. $f(z) = \dfrac{z-7}{z^2-2z-3} = \dfrac{-1}{z-3} + \dfrac{2}{z+1}$



$$f(z) = \frac{1/3}{1-\frac{z}{3}} + \frac{2}{1-(-z)} \quad , \ \left|\frac{z}{3}\right|<1 , \ |z|<1$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n + 2\sum_{n=0}^{\infty}(-z)^n$$

$$= \sum_{n=0}^{\infty}\left[\left(\frac{1}{3}\right)^{n+1} + 2\cdot(-1)^n\right]\cdot z^n$$

Radius of Convergence: $|z|<1$

#5.

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$$

$$e^{-t^2} = \sum_{n=0}^{\infty}\frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot t^{2n}}{n!}$$

$$\therefore \int_0^z e^{-t^2}\,dt = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot\int_0^z t^{2n}\,dt$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot\left.\frac{t^{2n+1}}{2n+1}\right|_0^z$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot\frac{z^{2n+1}}{2n+1}$$

$$\Rightarrow \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n\cdot z^{2n+1}}{(2n+1)\cdot n!}$$

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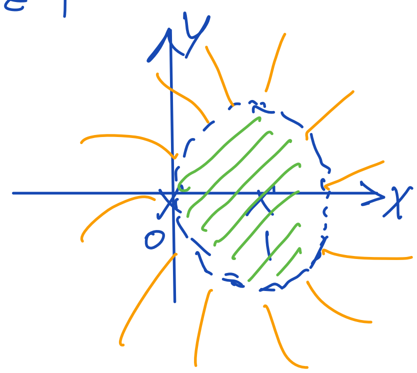
#6. $f(z) = \dfrac{7z-3}{z^2-z} = \dfrac{3}{z} + \dfrac{4}{z-1}$

$$\frac{3}{z} = \frac{3}{1+(z-1)}$$



(a) $0<|z-1|<1$

$$\frac{3}{1+(z-1)} = \frac{3}{1-[-(z-1)]}$$

$$= 3\cdot\sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

$$f(z) = \frac{4}{z-1} + 3\sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

(b) $|z-1|>1$

$$\frac{3}{1+(z-1)} = \frac{1}{z-1}\cdot\frac{3}{1+\frac{1}{z-1}} = \frac{3}{z-1}\cdot\frac{1}{1-\left(\frac{-1}{z-1}\right)}, \quad \left|\frac{1}{z-1}\right|<1$$

$$= \frac{3}{z-1}\cdot\sum_{n=0}^{\infty}\left(\frac{-1}{z-1}\right)^n = 3\cdot\sum_{n=0}^{\infty}(-1)^n\cdot\frac{1}{(z-1)^{n+1}}$$

$$f(z) = \frac{4}{z-1} + 3\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^{n+1}}$$