Complex Analysis 2022 Exam-3

D. C. Chang 2023/1/19

#1 $\int_{-\infty}^{\infty} \frac{\cos x + x\sin x}{x^2+1}\,dx$

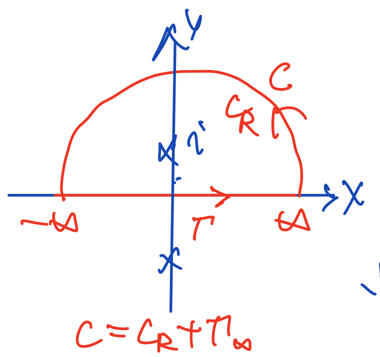
$= \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx + \int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1}\,dx$

$= I_1 + I_2$

① I_1:

Let $f(z) = \frac{e^{iz}}{z^2+1}$

upper half-plane pole at $z=i$



$\oint_C f(z)\,dz = I_1 + \int_{-\infty}^{\infty} \frac{i\sin x}{x^2+1}\,dx$

$= 2\pi i \cdot \mathrm{Res}(f(z), i)$

$\therefore I_1 = \mathrm{Re}\left\{2\pi i \cdot \mathrm{Res}(f(z), i)\right\}$

$= \mathrm{Re}\left\{2\pi i \cdot \frac{e^{iz}}{2z}\Big|_{z=i}\right\} = \frac{\pi}{e}$

② I_2:

Let $f(z) = \frac{ze^{iz}}{z^2+1}$

upper half-plane pole at $z=i$

$\oint_C f(z)\,dx = \int_{-\infty}^{\infty} \frac{x\cos x}{x^2+1}\,dx + i\cdot I_2$

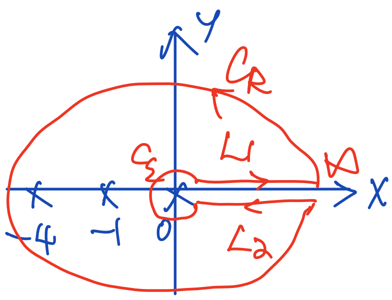
$\therefore I_2 = \mathrm{Im}\left\{\oint_C f(z)\,dz\right\}$

$= \mathrm{Im}\left\{2\pi i \cdot \mathrm{Res}(f(z), i)\right\}$

$= \mathrm{Im}\left\{2\pi i \cdot \frac{ze^{iz}}{2z}\Big|_{z=i}\right\} = \mathrm{Im}\left\{\frac{\pi i}{e}\right\} = \frac{\pi}{e}$

$\Rightarrow$ P.V. $\int_{-\infty}^{\infty} \frac{\cos x + x\sin x}{x^2+1}\,dx = I_1 + I_2 = \frac{2\pi}{e}$ #

#2. Let $f(z) = \dfrac{1}{\sqrt{z}\cdot(z^2+5z+4)} = \dfrac{1}{\sqrt{z}\,(z+1)(z+4)}$



$C = C_R + C_\varepsilon + L_1 + L_2$

$R \to \infty$

$\varepsilon \to 0$

$$\text{P.V.}\int_0^\infty \frac{1}{\sqrt{x}\,(x^2+5x+4)}\,dx$$

$$= \int_{L_1} f(x)\,dx$$

$$\because \oint_{C_R} f(z)\,dx = \oint_{C_\varepsilon} f(z)\,dz \to 0$$

$$\therefore \int_{L_1} f(x)\,dx + \int_{L_2} f(z)\,dz$$

$$= 2\pi i\cdot\left[\operatorname{Res}(f(z),-1) + \operatorname{Res}(f(z),-2)\right]$$

for L_2: $z = e^{i2\pi}\cdot x$

$$\int_{L_2} f(z)\,dz = \int_\infty^0 \frac{e^{i\pi}}{\sqrt{x}\,(x^2+5x+4)}\,dx$$

$$= \int_{L_1} f(z)\,dx$$

$$\operatorname{Res}(f(z),-1) = \left.\frac{z^{-\frac{1}{2}}}{2z+5}\right|_{z=-1} = \frac{e^{-\frac{\pi}{2}i}}{3} = -\frac{i}{3}$$

$$\operatorname{Res}(f(z),-4) = \frac{(-4)^{-\frac{1}{2}}}{-3} = \frac{e^{-\frac{\pi}{2}i}}{-6} = \frac{i}{6}$$

$$\therefore \text{P.V.}\int_0^\infty \frac{dx}{\sqrt{x}\,(x^2+5x+4)}$$

$$= 2\pi i\cdot\left(-\frac{i}{3} + \frac{i}{6}\right) = \frac{\pi}{3} \quad \#$$

#3. Let $f(z) = \dfrac{\pi \cot \pi z}{16z^2+16z+3}$

$16z^2+16z+3=0 \Rightarrow z = -\frac{1}{4}, -\frac{3}{4}$

$f(z)$ has simple poles at $z = -\frac{1}{4}, -\frac{3}{4}$

$$\Rightarrow \sum_{k=-\infty}^{\infty} \frac{1}{16k^2+16k+3} = -\left[\operatorname{Res}(f(z), -\tfrac{1}{4}) + \operatorname{Res}(f(z), -\tfrac{3}{4})\right]$$

$$\operatorname{Res}(f(z), -\tfrac{1}{4}) = \left.\frac{\pi \cot \pi z}{32z+16}\right|_{z=-\frac{1}{4}}$$

$$= \frac{\pi \cdot \cot(-\pi/4)}{8} = -\frac{\pi}{8}$$

$$\operatorname{Res}(f(z), -\tfrac{3}{4}) = \frac{\pi \cdot \cot(-3\pi/4)}{-24+16} = -\frac{\pi}{8}$$

$$\therefore \sum_{k=-\infty}^{\infty} \frac{1}{16k^2+16k+3} = \left(\sum_{k=0}^{\infty} + \sum_{k=-1}^{-\infty}\right) \frac{1}{16k^2+16k+3}$$

$$\sum_{k=-1}^{-\infty} \frac{1}{16k^2+16k+3} = \sum_{k=1}^{\infty} \frac{1}{16k^2-16k+3}, \quad p = k-1 \Downarrow k = p+1$$

$$= \sum_{p=0}^{\infty} \frac{1}{16p^2+32p+1-16p-1+3}$$

$$= \sum_{p=0}^{\infty} \frac{1}{16p^2+16p+3}$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{1}{16k^2+16k+3} = \frac{1}{2} \times (-1) \cdot \left(-\frac{\pi}{8} - \frac{\pi}{8}\right)$$

$$= \frac{\pi}{8} \quad \#$$

#4. $F(s)$ has simple poles at $s^2+6s+11=0$

$s=-3+\sqrt{2}i$, $s=-3-\sqrt{2}i$

$$f(t)=\mathcal{L}^{-1}\{F(s)\}=\operatorname{Res}\left(e^{st}\cdot\frac{s+4}{s^2+6s+11},-3+\sqrt{2}i\right)$$

$$+\operatorname{Res}\left(e^{st}\cdot\frac{s+4}{s^2+6s+11},-3-\sqrt{2}i\right)$$

$$\operatorname{Res}\left(e^{st}\cdot\frac{s+4}{s^2+6s+11},-3+\sqrt{2}i\right)$$

$$=\left.\frac{e^{st}\cdot(s+4)}{2s+6}\right|_{s=-3+\sqrt{2}i}=\frac{e^{(-3+\sqrt{2}i)t}}{2\sqrt{2}i}\cdot(1+\sqrt{2}i)$$

$$=\frac{\sqrt{2}-i}{2\sqrt{2}}\cdot e^{(-3+\sqrt{2}i)t}$$

$$\operatorname{Res}\left(e^{st}\cdot\frac{s+4}{s^2+6s+11},-3-\sqrt{2}i\right)$$

$$=\left.\frac{e^{st}\cdot(s+4)}{2s+6}\right|_{s=-3-\sqrt{2}i}=\frac{e^{(-3-\sqrt{2}i)t}}{-2\sqrt{2}i}\cdot(1-\sqrt{2}i)$$

$$=\frac{\sqrt{2}+i}{2\sqrt{2}}\cdot e^{(-3-\sqrt{2}i)t}$$

$$\Rightarrow f(t)=\frac{e^{-3t}}{2\sqrt{2}}\cdot\left[(\sqrt{2}-i)\cdot e^{i\sqrt{2}t}+(\sqrt{2}+i)\cdot e^{-i\cdot\sqrt{2}t}\right]$$

$$=\frac{e^{-3t}}{2\sqrt{2}}\Big[(\sqrt{2}-i)(\cos\sqrt{2}t+i\sin\sqrt{2}t)$$

$$+(\sqrt{2}+i)(\cos\sqrt{2}t-i\sin\sqrt{2}t)\Big]$$

$$=\frac{e^{-3t}}{2\sqrt{2}}\cdot\left[2\sqrt{2}\cos\sqrt{2}t+2\sin\sqrt{2}t\right]$$

$$=e^{-3t}\left(\cos\sqrt{2}t+\frac{1}{\sqrt{2}}\sin\sqrt{2}t\right)$$

#

#5 $F(\omega) = \dfrac{1}{(1-i\omega)^2}$

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(1-i\omega)^2}\, d\omega$$

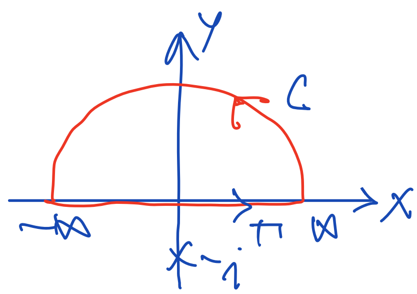
$$= -\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega+i)^2}\, d\omega$$

Let $g(z) = \dfrac{e^{itz}}{(z+i)^2}$,

$g(z)$ has a pole of order 2 at $z = -i$.

(a) For $t > 0$.

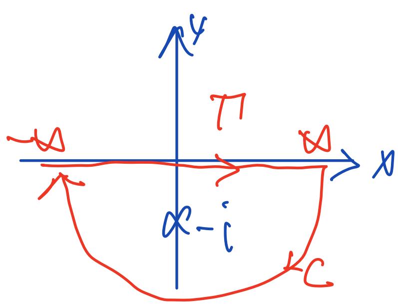
$$\oint_C g(z)\,dz = \int_\Gamma g(x)\,dx = 0$$



$C = C_R + \Gamma_\infty$

(b) For $t < 0$

$$\oint_C g(z)\,dz = \int_\Gamma g(x)\,dx = -2\pi i \cdot \operatorname{Res}(g(z), -i)$$



$$\operatorname{Res}(g(z), -i) = \lim_{z\to -i} \frac{d}{dz}(z+i)^2 g(z) = \lim_{z\to -i} it\cdot e^{itz} = i\cdot t e^{t}$$

$$\therefore \int_\Gamma g(x)\,dx = -2\pi i \cdot it e^{t} = 2\pi t\cdot e^{t}$$

$$f(t) = -\int_t g(x)\,dx$$
$$= -2\pi t \cdot e^{t}, \quad t<0$$

Thus,

$$f(t) = \mathcal{L}^{-1}\{F(\omega)\} = \begin{cases} 0, & t>0 \\ -2\pi t \cdot e^{t}, & t<0 \end{cases}$$