Complex Analysis 2023
Exam-1, Oct. 2023
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#1.

(a) $\cot(\pi+2i) = \dfrac{\cos(\pi+2i)}{\sin(\pi+2i)}$

$$\cos(\pi+2i) = \frac{e^{i\pi-2}+e^{-i\pi+2}}{2}$$

$$= -\frac{e^{-2}+e^{2}}{2} = -\cosh 2$$

$$\sin(\pi+2i) = \frac{e^{i\pi-2}-e^{-i\pi+2}}{2i}$$

$$= -\frac{e^{-2}-e^{2}}{2i} = -i\sinh 2$$

$$\therefore \cot(\pi+2i) = \underline{-i\coth 2} \quad ※$$

(b)

$$(1+\sqrt{3}i)^{3i} = e^{3i\cdot\ln(1+\sqrt{3}i)}$$

$$\ln(1+\sqrt{3}i) = \ln 2 + i\left(\frac{\pi}{3}+2n\pi\right), \quad n\in\mathbb{Z}$$

$$\therefore (1+\sqrt{3}i)^{3i} = e^{i\cdot 3\ln 2 - 3\left(\frac{\pi}{3}+2n\pi\right)}$$

$$= e^{-(2n+1)\pi + i\cdot 3\ln 2}$$

$$= \underline{e^{-(2n+1)\pi}\cdot(\cos 3\ln 2 + i\sin 3\ln 2)} \quad ※$$

$n\in\mathbb{Z}$

#2.

(a)

$$\sin z = \cos z$$

$$\frac{e^{iz}-e^{-iz}}{2i} = \frac{e^{iz}+e^{-iz}}{2}$$

$$\Rightarrow e^{2iz}-1 = i\cdot(e^{2iz}+1)$$

$$(e^{iz})^2 = \frac{1+i}{1-i} = i = e^{i(\frac{\pi}{2}+2n\pi)}, \quad n \in \mathbb{Z}$$

$$\therefore e^{iz} = e^{i(\frac{\pi}{4}+n\pi)} = e^{i\frac{4n+1}{4}\pi}$$

$$\underline{z = \frac{4n+1}{4}\pi}, \quad n \in \mathbb{Z}$$

(b)

$$\cos z = \bar{i}\sin z$$

$$\frac{e^{iz}+e^{-iz}}{2} = \frac{e^{iz}-e^{-iz}}{2}$$

$$\Rightarrow 2\cdot e^{-iz} = 0, \quad z = x+iy, \quad x, y \in \mathbb{R}$$

$$\Rightarrow e^{-y}(\cos x - i\sin x) = 0$$

$$\Rightarrow e^{-y} = 0, \quad y = \infty\ ?$$

$\therefore$ $\cos z = \bar{i}\sin z$ has no solution in $\mathbb{C}$. ※

#3.

(a)

$\sin w = z \quad , w = \sin^{-1} z$

$$\frac{e^{iw} - e^{-iw}}{2i} = z$$

$\Rightarrow e^{2iw} - 2i \cdot z \cdot e^{iw} - 1 = 0$

$e^{iw} = iz + (1-z^2)^{\frac{1}{2}}$

$\therefore w = -i \ln [iz + (1-z^2)^{\frac{1}{2}}]$

(b)

By the same way,

$\cos^{-1} z = -i \ln [z + i \cdot (1-z^2)^{\frac{1}{2}}]$

$\sin^{-1} z + \cos^{-1} z$

$= -i \cdot \ln [(iz + (1-z^2)^{\frac{1}{2}}) \cdot (z + i(1-z^2)^{\frac{1}{2}})]$

$= -i \ln (iz^2 - z(1-z^2)^{\frac{1}{2}} + z(1-z^2)^{\frac{1}{2}} + i(1-z^2))$

$= -i \ln i$

$= -i \ln \cdot e^{i(\frac{\pi}{2} + 2n\pi)} \quad , n \in \mathbb{Z}$

$= -i \cdot i (\frac{\pi}{2} + 2n\pi)$

$$= \frac{(4n+1)\pi}{2}$$

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#4.

(a) $f(z) = |z|^2$

$$f'(z)\Big|_{z=0} = \lim_{\Delta z \to 0} \frac{|0+\Delta z|^2 - |0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{|\Delta z|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \overline{\Delta z}$$

It is obvious that $\lim_{\Delta z \to 0} \overline{\Delta z} = 0$

That is, $f'(0) = 0$, f is differentiable at $z=0$.

(b) cosider that at any point $z = x_0 + iy_0$, $x_0 \neq y_0 \neq 0$

(i) If Δz approaches 0 along the x-axis, $\Delta y = 0$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{|x_0 + iy_0 + \Delta z|^2 - |x_0 + iy_0|^2}{\Delta z}$$

$$= \lim_{\Delta x \to 0} \frac{|(x_0+\Delta x) + iy_0|^2 - |x_0 + iy_0|^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x}$$

$$= 2x_0$$

(ii) If Δz approaches 0 along the y-axis, $\Delta x = 0$

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{|x_0 + i(y_0 + \Delta y)|^2 - |x_0 + iy_0|^2}{i \cdot \Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{(y_0 + \Delta y)^2 - y_0^2}{i \cdot \Delta y}$$

$$= -2i \cdot y_0$$

$\Rightarrow$ If $2x_0 = -2i \cdot y_0$, $x_0 = y_0 = 0$.

That is $f'(z)$ does not exist, except at $z=0$.

#5.

$z = x + iy$

$f(z) = u(x,y) + i\,v(x,y)$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{i\partial y} + i\cdot\frac{\partial v}{i\partial y} = \frac{\partial v}{\partial y} - i\cdot\frac{\partial u}{\partial y}$$

$$= 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0$$

① $\frac{\partial u}{\partial x} = 0 \Rightarrow u = g(y)$, independent of x

$\frac{\partial u}{\partial y} = g'(y) = 0 \quad \therefore g(y) = c$ (const.)

② $\frac{\partial v}{\partial x} = 0 \Rightarrow v = h(y)$, independent of x

$\frac{\partial v}{\partial y} = h'(y) = 0 \quad \therefore h(y) = c'$ (const)

$\Rightarrow$ $f(z) = u + i\cdot v = c + i\cdot c'$ is a const.

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#6.

$f(z) = u(r,\theta) + i v(r,\theta)$

If f is analytic,

$u_r = \frac{1}{r} v_\theta$, $v_r = -\frac{1}{r} u_\theta$

$\because u(r,\theta) = \frac{10r^2 - \sin 2\theta}{r^2}$

$\therefore$ ① $v_\theta = r \cdot \frac{\partial u}{\partial r}$

$= r \cdot \frac{20r \cdot r^2 - (10r^2 - \sin 2\theta) \cdot 2r}{r^4}$

$= \frac{20r^3 - 20r^3 + 2r\sin 2\theta}{r^3} = \frac{2\sin 2\theta}{r^2}$

$v = -\frac{1}{r^2}\cos 2\theta + g(r)$, $g(r)$ independent of θ

② $v_r = \frac{2\cos 2\theta}{r^3} + g'(r)$

$= -\frac{1}{r} \cdot \frac{-2\cos 2\theta}{r^2} = \frac{2\cos 2\theta}{r^3}$

$\therefore g'(r) = 0$, $\Rightarrow g(r) = C$, independent of r, θ

$\Rightarrow v = -\frac{1}{r^2}\cos 2\theta + C$, C: const.