Complex Analysis 2023

Exam #2 Solution

#1. Evaluate $\oint_C \left(\frac{1}{(z+1)^3} - \frac{5}{z+i} + 8 \right) dz$

$|z+i|=1 \Rightarrow z+i = e^{i\theta}$ $dz = ie^{i\theta}d\theta$ $0 \le \theta \le 2\pi$

$z+1 = z+i+1-i = e^{i\theta} + \alpha$, $\alpha = 1-i$

$$\oint_C (z+1)^{-3} dz = \int_0^{2\pi} (e^{i\theta}+\alpha)^{-3} \cdot i \cdot e^{i\theta} d\theta$$

$$= -\frac{1}{2}(e^{i\theta}+\alpha)^{-2}\Big|_0^{2\pi}$$

$$= -\frac{1}{2}\left[(e^{i2\pi}+\alpha)^{-2} - (1+\alpha)^{-2}\right]$$

$$= 0$$

$$\oint_C \left(\frac{-5}{z+i}+8\right)dz = \int_0^{2\pi} (-5e^{-i\theta}+8)\, i \cdot e^{i\theta} d\theta$$

$$= \int_0^{2\pi} (-5i + 8i \cdot e^{i\theta}) d\theta$$

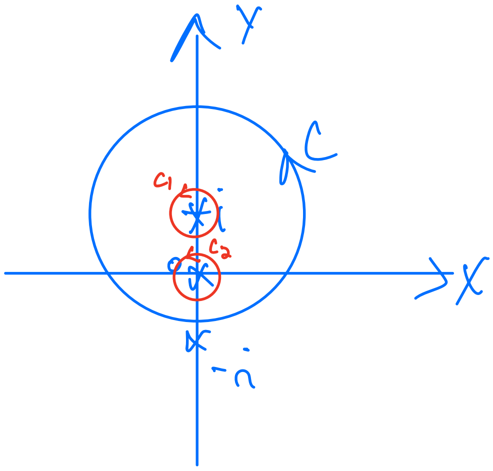
$$= -5i \cdot \theta\Big|_0^{2\pi} + 8i \cdot \frac{1}{i} e^{i\theta}\Big|_0^{2\pi}$$

$$= -10\pi i$$

Ans: $0 + (-10\pi i) = \underline{-10\pi i}$ #

#2.

(a) $z^2(z^2+1)=0 \Rightarrow z=0, +i, -i$



$$\oint_C \frac{dz}{z^2(z^2+1)}$$

$$= \oint_{C_1} \frac{\frac{1}{z^2(z+i)}dz}{z-i}$$

$$+ \oint_{C_2} \frac{1/(z^2+1)}{z^2}dz$$

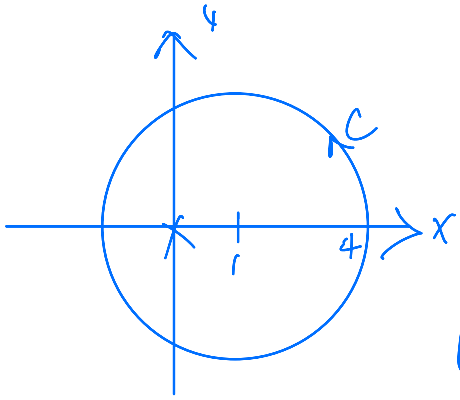
$$= 2\pi i \cdot \frac{1}{z^2(z+i)}\Big|_{z=i}$$

$$+ 2\pi i \cdot \left(\frac{1}{z^2+1}\right)'\Big|_{z=0}$$

$$= \frac{2\pi i}{-1 \cdot 2i} + 2\pi i \cdot \frac{2z}{z^2+1}\Big|_{z=0}$$

$$= -\pi \quad \#$$

#2. (b)



$$\oint_C \frac{e^{-z}\sin z}{z^3}\,dz$$

$$= \frac{2\pi i}{2!}\left(e^{-z}\sin z\right)''\Big|_{z=0}$$

$$\left(e^{-z}\sin z\right)' = -e^{-z}\sin z + e^{-z}\cos z$$

$$\left(e^{-z}\sin z\right)'' = \cancel{e^{-z}\sin z} - e^{-z}\cos z - e^{-z}\cos z \cancel{- e^{-z}\sin z}$$

$$= -2e^{-z}\cos z$$

$$\left(e^{-z}\sin z\right)''\Big|_{z=0} = -2\cos 0 = -2$$

$$\therefore \oint_C \frac{e^{-z}\sin z}{z^3}\,dz = \pi i\cdot(-2) = -2\pi i$$

#3 (a)

$$\because \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{for } |z|<1$$

$$\therefore \sum_{n=1}^{\infty} n\cdot z^{n-1} = \frac{d}{dz}\frac{1}{1-z} = \frac{1}{(1-z)^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} n\cdot z^n = \frac{z}{(1-z)^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} n^2\cdot z^{n-1} = \frac{d}{dz}\left(\frac{z}{(1-z)^2}\right) = \frac{1+z}{(1-z)^3}$$

$$\therefore \sum_{n=1}^{\infty} n^2 z^{n-1} = \sum_{n=0}^{\infty}(n+1)^2 z^n$$

$$= \frac{1+z}{(1-z)^3}$$

#3(b)

By root test,

$\sum_{n=0}^{\infty} C_n z^n$ has R.O.C R

Then, $R = \frac{1}{L}$ and $L = \lim_{n\to\infty} \sqrt[n]{|C_n|}$

For the power series, $\sum_{n=0}^{\infty} C_n^2 z^n$,

$$ROC = \frac{1}{L'}, \quad L' = \lim_{n\to\infty} \sqrt[n]{|C_n^2|}$$

$$= \lim_{n\to\infty} \sqrt[n]{|C_n|^2}$$

$$= \lim_{n\to\infty} \left(\sqrt[n]{|C_n|}\right)^2$$

$$= L^2$$

i.e, $ROC = \frac{1}{L^2} = \left(\frac{1}{L}\right)^2$

$$= R^2$$

$\#\#$

#4.

$$\frac{1}{1-z} = \frac{1}{1-i} \cdot \frac{1}{1-\frac{z-i}{1-i}}, \quad |z-i| < \underset{\sqrt{2}}{\underbrace{|1-i|}}$$

$$= \frac{1}{1-i} \cdot \sum_{n=0}^{\infty} \left(\frac{z-i}{1-i}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}$$

$\#\#$

#5

(a) $e^{iz} = \cos z + i\sin z$

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}$$

$$\cos^3 z = \frac{1}{8}(e^{iz}+e^{-iz})^3$$

$$= \frac{1}{8}(e^{i3z}+3e^{iz}+3e^{-iz}+e^{-i3z})$$

$$= \frac{1}{4}\cdot\frac{1}{2}(e^{i3z}+e^{-i3z})+\frac{3}{4}\cdot\frac{1}{2}(e^{iz}+e^{-iz})$$

$$= \frac{1}{4}\cos 3z + \frac{3}{4}\cos z$$

$$\therefore 4\cos^3 z = \cos 3z + 3\cos z$$

(b) $\because \cos z = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{z^{2n}}{(2n)!}$

$$\therefore \cos^3 z = \frac{1}{4}(\cos 3z + 3\cos z)$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n \frac{(3z)^{2n}+3\cdot z^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n \cdot \frac{3^{2n}+3}{4\cdot(2n)!}\cdot z^{2n}$$

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#6. $$\frac{1}{z^2-5z+6} = \frac{1}{2-z} - \frac{1}{3-z}$$

① for $|z|<2$

$$\frac{1}{z^2-5z+6} = \frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}} - \frac{1}{3}\cdot\frac{1}{1-\frac{z}{3}}$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{2}\left(\frac{z}{2}\right)^n - \frac{1}{3}\left(\frac{z}{3}\right)^n\right]$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}}\right)z^n$$ ※

② for $2<|z|<3$

$$\frac{1}{z^2-5z+6} = \frac{-1}{z}\cdot\frac{1}{1-\frac{2}{z}} - \frac{1}{3}\cdot\frac{1}{1-\frac{z}{3}}$$

$$= \sum_{n=0}^{\infty}\left[\frac{-1}{z}\cdot\left(\frac{2}{z}\right)^n - \frac{1}{3}\cdot\left(\frac{z}{3}\right)^n\right]$$

$$= \sum_{n=0}^{\infty}\left(\frac{-2^n}{z^{n+1}} - \frac{z^n}{3^{n+1}}\right)$$ ※

③ for $|z|>3$

$$\frac{1}{z^2-5z+6} = \frac{-1}{z}\cdot\frac{1}{1-\frac{2}{z}} + \frac{1}{z}\cdot\frac{1}{1-\frac{3}{z}}$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\left[\left(\frac{2}{z}\right)^n + \left(\frac{3}{z}\right)^n\right]$$

$$= -\sum_{n=0}^{\infty}(2^n+3^n)z^{-n-1}$$ ※