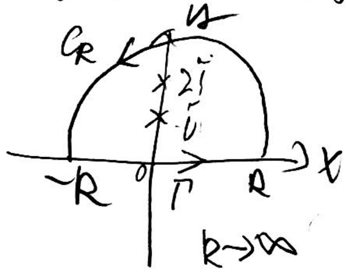
Complex Analysis 2023 Exam-3 DCChang./NCU

#1. $f(z)=\dfrac{z^2}{(z^2+4)(z^2+1)^2}$

$\int_0^\infty f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx$. $\quad C=\Gamma+C_R$



$\int_{-\infty}^{\infty} f(x)\,dx = \oint_C f(z)\,dz - \int_{C_R} f(z)\,dz$

$\because \int_{C_R} f(z)\,dz \to 0$

$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\cdot\left(Res(f,2i)+Res(f,i)\right)$

$$Res(f,i)=\lim_{z\to i}\frac{d}{dz}\left(\frac{z^2}{(z^2+4)(z+i)^2}\right)$$

$$=\left.\frac{2z(z^2+4)(z+i)^2-z^2[2z(z+i)^2+2(z+i)(z^2+4)]}{(z^2+4)^2(z+i)^4}\right|_{z=i}$$

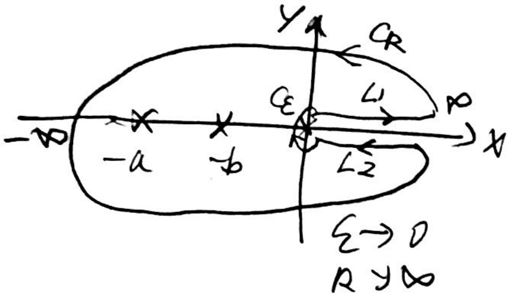
$$=\frac{2i\times3\times(-4)+2i\cdot(-4)+4i\times3}{9\times2^4}$$

$$=\frac{-24i-8i+12i}{9\times2^4}=\frac{-5i}{36}$$

$$Res(f,2i)=\lim_{z\to2i}\frac{z^2}{(z+2i)(z^2+1)^2}=\frac{-4}{4i\cdot9}=\frac{i}{9}$$

$$\therefore \int_0^\infty f(x)\,dx=\pi i\cdot\left(-\frac{5}{36}+\frac{1}{9}\right)i=\underline{\frac{\pi}{36}}\ \#$$

#2. $f(z) = \dfrac{z^{\frac{1}{3}}}{(z+a)(z+b)}$



$C = C_\varepsilon + C_R + L_1 + L_2$

$\oint_C = \int_{C_R} + C_\varepsilon + L_1 + L_2$

$\because \int_{C_R} = \int_{C_\varepsilon} = 0 \quad (\because \deg(Q) \geq \deg(P) + 2)$

$\therefore \int_{L_1 + L_2} = 2\pi i \left[Res(f, -a) + Res(-b) \right]$

$\int_0^\infty f(x)\,dx = \int_{L_1} f(z)\,dz$

For L_2: $z = e^{i2\pi} x$, $dz = dx$

$$\int_{L_2} f(z)\,dz = e^{i\frac{2\pi}{3}} \int_\infty^0 \frac{x^{\frac{1}{3}}\,dx}{(x+a)(x+b)}$$

$$= -e^{i\frac{2\pi}{3}} \cdot \int_0^\infty f(x)\,dx$$

$$\Rightarrow \left(1 - e^{i\frac{2\pi}{3}}\right) \cdot \int_0^\infty f(x)\,dx = 2\pi i \cdot \left[Res(f, -a) + Res(f, -b) \right]$$

$$Res(f, -a) = \lim_{z \to -a} \frac{z^{1/3}}{z+b} = \frac{\sqrt[3]{a} \cdot (-1)^{1/3}}{b-a} = \frac{\sqrt[3]{a}}{b-a} e^{\frac{i\pi}{3}}$$

$$Res(f, -b) = \lim_{z \to -b} \frac{z^{1/3}}{z+a} = \frac{\sqrt[3]{b}}{a-b} e^{i\frac{\pi}{3}}$$

$$\therefore \int_0^\infty f(x)\,dx = \frac{1}{1 - e^{i\frac{2\pi}{3}}} \cdot \frac{\sqrt[3]{b} - \sqrt[3]{a}}{a-b} \cdot e^{i\frac{\pi}{3}} \cdot 2\pi i$$

$$= \frac{1}{e^{-i\frac{\pi}{3}} - e^{i\frac{\pi}{3}}} \cdot \frac{\sqrt[3]{b} - \sqrt[3]{a}}{a-b} \cdot 2\pi i$$

$$= \frac{\sqrt[3]{b} - \sqrt[3]{a}}{a-b} \cdot \frac{2\pi i}{\frac{1}{2} - \frac{\sqrt{3}}{2} i - \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)}$$

$$= \frac{2\pi}{\sqrt{3}} \times \frac{\sqrt[3]{a} - \sqrt[3]{b}}{a-b}$$

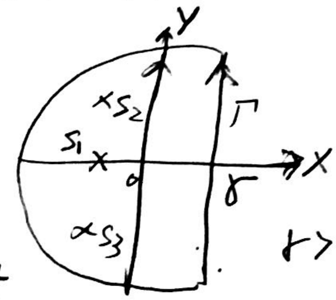
$\#$

#3. $s^2+2s+5=0 \Rightarrow s=-1\pm 2i$

F(s) has simple poles at $s_1=-1,\ s_2=-1+2i,\ s_3=-1-2i$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma-j\infty}^{\gamma+j\infty} e^{st}F(s)\,ds \quad , \gamma>-1.$$

$$=\sum_{k=1}^{3} \mathrm{Res}[e^{st}\cdot F(s), s_k].$$



$\gamma > -1$.

$\mathrm{Res}(e^{st}F(s), 1)$

$$= \frac{e^{st}(2s-2)}{s^2+2s+5}\Big|_{s=-1} = \frac{e^{-t}\cdot(-4)}{4} = -e^{-t}$$

$\mathrm{Res}(e^{st}F(s), -1+2i)$

$$= \frac{e^{st}\cdot(2s-2)}{(s+1)[s-(-1-2i)]}\Big|_{s=-1+2i} = \frac{e^{-t}\cdot e^{i2t}\cdot(-4+4i)}{2i\times 4i}$$

$$= \frac{1-i}{2}\cdot e^{-t}\cdot e^{i2t}$$

$\mathrm{Res}(e^{st}F(s), -1-2i)$

$$= \frac{e^{st}\cdot(2s-2)}{(s+1)[s-(-1+2i)]}\Big|_{s=-1-2i} = \frac{e^{-t}\cdot e^{-i2t}\cdot(-4-4i)}{(-2i)\times(-4i)}$$

$$= \frac{1+i}{2}\cdot e^{-t}\cdot e^{-i2t}$$

$$\therefore \mathcal{L}^{-1}[F(s)] = -e^{-t} + e^{-t}\cdot\left[\frac{1-i}{2}e^{i2t} + \frac{1+i}{2}e^{-i2t}\right]$$

$$= e^{-t}\left(-1 + \frac{e^{i2t}+e^{-i2t}}{2} - i\frac{e^{i2t}-e^{-i2t}}{2}\right)$$

$$= \underline{e^{-t}\cdot(-1+\cos 2t + \sin 2t)}$$ #

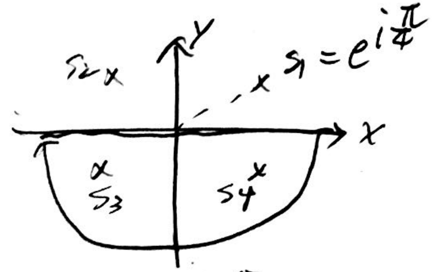
#4. $F(w) = \mathcal{F}\{f(t)\}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{t e^{-iwt}}{t^4+1}\,dt \quad , \quad g(z) = \frac{z e^{-iwz}}{z^4+1}$$

① For $w > 0$,

$$I_1 = \int_{-\infty}^{\infty} g(x)\,dx$$

$$= -2\pi i \cdot \Big[\mathrm{Res}(g(z), S_3) + \mathrm{Res}(g(z), S_4)\Big]$$



$$S_3 = e^{i\frac{5}{4}\pi} = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$S_4 = e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

$$\mathrm{Res}(g(z), S_3) = \frac{z e^{-iwz}}{4z^3}\Big|_{z=e^{i\frac{5}{4}\pi}}$$

$$= \frac{1}{4} \cdot e^{-\frac{5}{2}\pi} \cdot e^{-iw(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)}$$

$$= \frac{-i}{4} \cdot e^{-w/\sqrt{2}} \cdot e^{iw/\sqrt{2}}$$

$$\mathrm{Res}(g(z), S_4) = \frac{1}{4} z^{-2} e^{-iwz}\Big|_{z=e^{-i\frac{\pi}{4}}}$$

$$= \frac{1}{4} e^{i\frac{\pi}{2}} \cdot e^{-iw(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)}$$

$$= \frac{i}{4} e^{-w/\sqrt{2}} \cdot e^{-iw/\sqrt{2}}$$

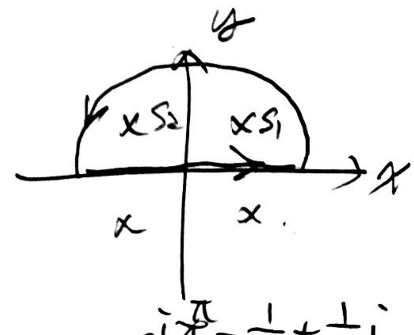
$$I_1 = -2\pi i \cdot \frac{i}{4} \cdot e^{-w/\sqrt{2}}\left(-e^{iw/\sqrt{2}} + e^{-iw/\sqrt{2}}\right)$$

$$= \frac{\pi}{2} \cdot e^{-w/\sqrt{2}}\left(-2i \cdot \sin\frac{w}{\sqrt{2}}\right) = -\pi i \cdot e^{-w/\sqrt{2}} \sin\frac{w}{\sqrt{2}}$$

$$F(w) = \frac{1}{2\pi} I_1 = \underline{-\frac{i}{2} \cdot e^{-w/\sqrt{2}} \cdot \sin\frac{w}{\sqrt{2}}, \; w > 0}$$

② For $w < 0$

$$I_2 = \int_{-\infty}^{\infty} g(z)\,dz = 2\pi i\Big[\mathrm{Res}(g(z), S_1) + \mathrm{Res}(g(z), S_2)\Big]$$



$$\mathrm{Res}(g(z), S_1) = \frac{1}{4} z^{-2} \cdot e^{-iwz}\Big|_{z=e^{i\frac{\pi}{4}}}$$

$$= \frac{1}{4} e^{-i\frac{\pi}{2}} \cdot e^{-iw(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)}$$

$$= \frac{-i}{4} \cdot e^{w/\sqrt{2}}\, e^{-iw/\sqrt{2}}$$

$$S_1 = e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$S_2 = e^{i\frac{3\pi}{4}} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$\mathrm{Res}(g(z), s_2) = \frac{1}{4} z^{-2} \cdot e^{-iwz} \Big|_{z = e^{i\frac{3}{4}\pi}}$$

$$= \frac{1}{4} e^{-i\frac{3}{2}\pi} e^{-iw\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)}$$

$$= \frac{i}{4} \cdot e^{w/\sqrt{2}} \cdot e^{iw/\sqrt{2}}$$

$$I_2 = 2\pi i \cdot \frac{i}{4} \cdot e^{w/\sqrt{2}} \left(-e^{-iw/\sqrt{2}} + e^{iw/\sqrt{2}}\right)$$

$$= \frac{-\pi}{2} e^{w/\sqrt{2}} \cdot 2i \sin\left(\frac{w}{\sqrt{2}}\right)$$

$$= -\pi i \cdot e^{w/\sqrt{2}} \cdot \sin\left(\frac{w}{\sqrt{2}}\right)$$

$$F(w) = \frac{1}{2\pi} \cdot I_2 = -\frac{i}{2} \cdot e^{w/\sqrt{2}} \cdot \sin\left(\frac{w}{\sqrt{2}}\right), \quad w < 0 \quad //$$

or Combining ① and ②:

$$F(w) = -\frac{i}{2} \cdot e^{-|w|/\sqrt{2}} \cdot \sin\left(\frac{w}{\sqrt{2}}\right) \quad //$$