Complex Analysis 2024 Exam 1

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1.

(a)

$$z^2 - 1 = e^{\mathrm{Log}(z^2-1)} = e^{i\pi/2} = i$$
$$\Longleftrightarrow z^2 = 1 + i$$
$$\Longleftrightarrow z = (1+i)^{1/2} = \sqrt[4]{2}e^{i\pi/8}, \sqrt[4]{2}e^{i9\pi/8}$$

(b)

$$e^z = \frac{-1+\sqrt{-3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\begin{aligned} z &= \log\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) \\ &= \mathrm{Log}\left|-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right| + i\arg\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right) \\ &= i\left(\pm\frac{2\pi}{3} + 2k\pi\right), \quad k = 0, \pm 1, \ldots \end{aligned}$$

2.

Set $w = \sec^{-1} z$. Then

$$z = \sec w = \frac{2}{e^{iw} + e^{-iw}}$$

$$ze^{iw} + ze^{-iw} = 2$$

$$ze^{2iw} - 2e^{iw} + z = 0$$

$$e^{iw} = \frac{2 + (4 - 4z^2)^{1/2}}{2z} \quad \text{(quadratic formula)}$$

$$e^{iw} = \frac{1}{z} + \left(\frac{1}{z^2} - 1\right)^{1/2}$$

$$iw = \log\left[\frac{1}{z} + \left(\frac{1}{z^2} - 1\right)^{1/2}\right]$$

$$\sec^{-1} z = w = -i\log\left[\frac{1}{z} + \left(\frac{1}{z^2} - 1\right)^{1/2}\right].$$

3.

(a)

$$\begin{aligned} z = \tan w &= -i\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} \Rightarrow \\ e^{2iw} &= \frac{1+iz}{1-iz} \Rightarrow \\ w &= \frac{-i}{2}\log\left(\frac{1+iz}{1-iz}\right) \\ &= \frac{i}{2}\log\left(\frac{1-iz}{1+iz}\right) = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right) \end{aligned}$$

(b)

choose a branch of the logarithm.

$$\begin{aligned} \frac{d}{dz}\left(\tan^{-1} z\right) &= \frac{d}{dz}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right] \\ &= \frac{i}{2}\frac{i-z}{i+z}\frac{2i}{(i-z)^2} \\ &= \frac{1}{1+z^2}, \quad z \neq \pm i \end{aligned}$$

4.

$u = ax^2 + bxy + cy^2$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

① $\frac{\partial v}{\partial y} = 2ax + by$

$\Rightarrow v = 2axy + \frac{1}{2}by^2 + A(x)$

② $-\frac{\partial v}{\partial x} = -2ay - A'(x)$

$= bx + 2cy$

$\therefore c = -a, \quad A'(x) = -bx$

$\Rightarrow A(x) = -\frac{1}{2}bx^2 + d$

We have that

$v = 2axy + \frac{1}{2}by^2 - \frac{1}{2}bx^2 + d$

5.

Case 1, $z = 0$.

$$\lim_{\Delta z \to 0} \frac{|0 + \Delta z| - |0|}{\Delta z} = \lim_{\Delta z \to 0} \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta x + i\Delta y} = \begin{cases} \pm 1, & \text{if } \Delta z = \Delta x \\ -i, & \text{if } \Delta z = \pm i\Delta y \end{cases}$$

Case 2, $z \neq 0$.

$$\begin{aligned} &\lim_{\Delta z \to 0} \frac{|z + \Delta z| - |z|}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\sqrt{(x + \Delta x)^2 + (y + \Delta y)^2} - \sqrt{x^2 + y^2}}{\Delta x + i\Delta y} \\ &= \lim_{\Delta z \to 0} \frac{(x + \Delta x)^2 + (y + \Delta y)^2 - (x^2 + y^2)}{(\Delta x + i\Delta y)(\sqrt{(x + \Delta x)^2 + (y + \Delta y)^2} + \sqrt{x^2 + y^2})} \\ &= \lim_{\Delta z \to 0} \frac{2x\Delta x + (\Delta x)^2 + 2y\Delta y + (\Delta y)^2}{(\Delta x + i\Delta y)(\sqrt{(x + \Delta x)^2 + (y + \Delta y)^2} + \sqrt{x^2 + y^2})} \\ &= \begin{cases} \dfrac{x}{\sqrt{x^2 + y^2}}, & \text{if } \Delta z = \Delta x,\ z \neq 0 \\ \dfrac{y}{i\sqrt{x^2 + y^2}}, & \text{if } \Delta z = i\Delta y,\ z \neq 0 \end{cases} \end{aligned}$$

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(a)

$$\left(\frac{2i}{1 + i}\right)^{1/6} = (1+i)^{1/6} = \sqrt[12]{2} \exp\left(i\frac{\pi/4 + 2k\pi}{6}\right), k = 0, 1, 2, 3, 4, 5$$

(b)

$$1^{1/5} = \exp\left(i\frac{2k\pi}{5}\right), k = 0, 1, 2, 3, 4$$