Complex Analysis 2024   Exam 2

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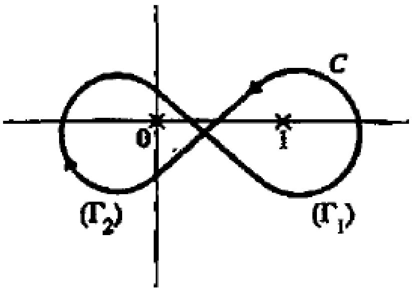
1.

$$\int_C \frac{2z+1}{z(z-1)^2}\,dz = \int_{\Gamma_1} \frac{(2z+1)/z}{(z-1)^2}\,dz + \int_{\Gamma_2} \frac{(2z+1)/(z-1)^2}{z}\,dz,$$

$$\frac{2\pi i}{1!}\frac{d}{dz}\left(\frac{2z+1}{z}\right)\Bigg|_{z=1} - 2\pi i\,\frac{2z+1}{(z-1)^2}\Bigg|_{z=0} = -2\pi i - 2\pi i = -4\pi i.$$



2.

Let $g(\zeta) = \zeta^2 - \zeta + 2$

Then $G(1) = 2\pi i g(1) = 4\pi i$

$G'(i) = 2\pi i g'(i) = -4\pi - 2\pi i$

$G''(-i) = 2\pi i g''(-i) = 4\pi i$

3.

The circle $|z| = 1$ has a parametrization $z(t) = e^{it}$, $0 \le t \le 2\pi$. Thus, $z'(t) = ie^{it}$.

$$\begin{aligned}\oint_C \mathrm{Re}(z)dz &= \int_0^{2\pi} \mathrm{Re}(e^{it}) ie^{it}\,dt \\ &= i\int_0^{2\pi} \cos t(\cos t + i\sin t)dt \\ &= i\int_0^{2\pi} (\cos^2 t + i\sin t\cos t)dt \\ &= i\int_0^{2\pi} \frac{1+\cos 2t}{2}dt - \int_0^{2\pi} \sin t\cos t\,dt \\ &= i\left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} - \left[\frac{\sin^2 t}{2}\right]_0^{2\pi} \\ &= \pi i.\end{aligned}$$

4.

(a)

$$\sum_{k=1}^{\infty}(-1)^k\left(\frac{1+2i}{2}\right)^k(z+2i)^k$$

$$a_n=(-1)^n\left(\frac{1+2i}{2}\right)^n$$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\frac{1+2i}{2}\right|^n}=\lim_{n\to\infty}\left|\frac{1+2i}{2}\right|$$

$$=\left|\frac{1+2i}{2}\right|=\frac{\sqrt{5}}{2}$$

The radius of convergence is $R=\frac{2}{\sqrt{5}}$ and the circle of convergence is $|z+2i|=\frac{2}{\sqrt{5}}$

(b)

$$\sum_{k=1}^{\infty}\frac{k!}{(2k)^k}z^{3k}=\sum_{k=1}^{\infty}\frac{k!}{(2k)^k}(z^3)^k$$

$$a_n=\frac{n!}{(2n)^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{2^{n+1}(n+1)^{n+1}}}{\frac{n!}{2^n n^n}}\right|=\lim_{n\to\infty}\frac{(n+1)(2^n n^n)}{2^{n+1}(n+1)^{n+1}}$$

$$=\lim_{n\to\infty}\frac{1}{2}\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{n+1}\right)^2=\frac{e^{-1}}{2}$$

The radius of convergence is $R = 2e$ and the circle of convergence is

$$|z^3|=2e\Rightarrow|z|=\sqrt[3]{2e}$$

5.

$$f(z)=\sum_{k=0}^{\infty}\frac{f_{(0)}^{(k)}}{k!}z^k$$

$$f(z)=e^{\frac{1}{1+z}} \qquad f(0)=e$$

$$f'(z)=\frac{-1}{(1+z)^2}e^{\frac{1}{1+z}} \qquad f'(0)=-e$$

$$f''(z)=\left[\frac{2}{(1+z)^3}+\frac{1}{(1+z)^4}\right]e^{\frac{1}{1+z}} \qquad f''(0)=3e$$

$$\Rightarrow f(z)=e-ez+-\frac{3}{2!}ez^2+\ldots$$

6.

(a)

$$f(z)=\frac{1}{z}\cdot\left(7-\frac{4}{1-z}\right)= \frac{3}{z}-4-4z-4z^2-\cdots$$

(b)

$$f(z)=\frac{7z-3}{z(z-1)} \qquad 0<|z-1|<1$$

$$f(z)=\frac{3}{z}+\frac{4}{z-1}=\frac{3}{1+z-1}+\frac{4}{z-1}$$

$$=3[1-(z-1)+(z-1)^2-(z-1)^3+\ldots]+\frac{4}{z-1}$$

The series (in brackets) converges for $|z-1|<1$

since $z-1\neq 0,\ z\neq 1,\ f(z)$ is valid for $0<|z-1|<1$

$$f(z)=3-3(z-1)+3(z-1)^2-3(z-1)^3+\ldots\frac{4}{z-1}$$