Complex Analysis #3 2024

1.

$$\int_0^{2\pi} \frac{1}{\cos\theta + 2\sin\theta + 3} d\theta$$

$$= \oint_C \frac{1}{\left(\frac{z+z^{-1}}{2}\right) + 2\left(\frac{z-z^{-1}}{2i}\right) + 3} \frac{dz}{iz},\ C: |z| = 1$$

$$= \frac{2}{i}\oint_C \frac{1}{z+z^{-1} - 2i(z-z^{-1}) + 6} \frac{dz}{z} = \frac{2}{i}\oint_C \frac{1}{(1-2i)z^2 + 6z + (1+2i)} dz$$

$$= \frac{2}{i(1-2i)}\oint_C \frac{1}{z^2 \frac{6}{1-2i} z + \frac{1+2i}{1-2i}} dz = \frac{2}{i(1-2i)}\oint_C \frac{1}{z^2 + \left(\frac{6}{5} + \frac{12i}{5}\right)z - \frac{3}{5} + \frac{4}{5}i} dz$$

$f(z) = \dfrac{1}{z^2 + \left(\frac{6}{5} + \frac{12i}{5}\right)z - \frac{3}{5} + \frac{4}{5}i} = \dfrac{1}{(z+1+2i)\left(z + \frac{1}{5} + \frac{2}{5}i\right)}$ has simple poles at $z = -1-2i$

and $z = -\frac{1}{5} - \frac{2}{5}i$. Only the pole $z = -\frac{1}{5} - \frac{2}{5}i$ lies within the unit circle.

$$\oint_C f(z)dz = 2\pi i \operatorname{Res}\left(f(z), -\frac{1}{5} - \frac{2}{5}i\right)$$

$$\operatorname{Res}\left(f(z), -\frac{1}{5} - \frac{2}{5}i\right) = \operatorname*{limit}_{z\to -\frac{1}{5}-\frac{2}{5}i} \frac{1}{z+1+2i} = \frac{1}{4} - \frac{i}{2}$$

$$\int_0^{2\pi} \frac{1}{\cos\theta + 2\sin\theta + 3} d\theta = \frac{2}{i(1-2i)} \cdot 2\pi i \cdot \left(\frac{1}{4} - \frac{i}{2}\right) = \pi \frac{(1-2i)}{1-2i} = \pi$$

2.

$$\int_{-\infty}^{\infty} \frac{x\cos x}{x^2-3x+2}dx = \pi(\sin 1 - 2\sin 2)$$

$f(z) = \dfrac{z}{z^2-3z+2} = \dfrac{z}{(z-1)(z-2)}$ has simple poles at $z=1$ and $z=2$, both on the x-axis.

We then consider:

$\oint_C \dfrac{ze^{iz}}{z^2-3z+2}dz$, where C shown below is indented at $z=1$ and at $z=2$

$$\oint_C \frac{ze^{iz}}{z^2-3z+2}dz = 0$$

Adopting a condensed notation:

$$\oint_C = \int_{CR} + \int_{-R}^{1-r_1} + \int_{-C_{r_1}} + \int_{1+r_1}^{2-r_2} + \int_{-C_{r_2}} + \int_{2+r_2}^{R} = 0$$

If we take the limits as $R \to \infty, r_1 \to 0, r_2 \to 0$,

$$\text{P.V. } \int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2-3x+2}dx =$$

$$\pi\,\text{Res}\left(f(z)e^{iz},1\right) - \pi i\,\text{Res}\left(f(z)e^{iz},2\right) = 0$$

$$\text{Res}\left(f(z)e^{iz},1\right) = \underset{z\to 1}{\text{limit}} \frac{ze^{iz}}{z-2} = \frac{e^i}{-1} = -e^i$$

$$\text{Res}\left(f(z)e^{iz},2\right) = \underset{z\to 2}{\text{limit}} \frac{ze^{iz}}{z-1} = \frac{2e^{2i}}{1} = 2e^{2i}$$

Therefore P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2-3x+2}dx = \pi i\left(2e^{2i} - e^i\right)$

$$= \pi i(2\cos 2 + 2i\sin 2 - \cos 1 - i\sin 1)$$

$$= \pi(\sin 1 - 2\sin 2) + i\pi(2\cos 2 - \cos 1)$$

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 - 3x + 2} dx = \int_{-\infty}^{\infty} \frac{x\cos x}{x^2 - 3x + 2} dx + i\int_{-\infty}^{\infty} \frac{x\sin x}{x^2 - 3x + 2} dx$$

Equating real and imaginary parts, we get:

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2 - 3x + 2} dx = \pi(2\cos 2 - \cos 1)$$

and

$$\int_{-\infty}^{\infty} \frac{x\cos x}{x^2 - 3x + 2} dx = \pi(\sin 1 - 2\sin 2)$$

3.

$$\int_0^{\infty} \frac{x^{\frac{1}{3}}}{(x+1)^2} dx = \frac{2\pi}{3\sqrt{3}}$$

we form the integral $\oint_C \frac{z^{\frac{1}{3}}}{(z+1)^2}$, where C is the closed contour shown below

$f(z) = \frac{z^{\frac{1}{3}}}{(z+1)^2}$ is single-valued and analytic on and within C, except at $z = -1$.

Hence, we can write

$$\oint_C f(z)dz = 2\pi i \operatorname{Res}\big(f(z), -1\big)$$

or

$$\int_{CR} + \int_{ED} + \int_{C_r} + \int_{AB} = 2\pi i \operatorname{Res}\big(f(z), -1\big)$$

on AB, $z = xe^{oi}$ and on ED, $z = xe^{2\pi i}$, so that

$$\int_{ED} = \int_R^z \frac{\left(xe^{2\pi i}\right)^{\frac{1}{3}}}{\left(xe^{2\pi i}+1\right)}\left(e^{i2\pi}dx\right) = \int_R^z \frac{e^{2\pi\frac{i}{3}}x^{\frac{1}{3}}}{(x+1)^2}dx$$

$$\int_{ED} = -e^{2\pi\frac{i}{3}}\int_r^R \frac{x^{\frac{1}{3}}}{(x+1)^2}dx$$

$$\int_{AB} = \int_r^R \frac{(xe^{io})^{\frac{1}{3}}}{(xe^{io}+1)^2}(e^{io}dx) = \int_r^R \frac{x^{\frac{1}{3}}}{(x+1)^2}dx$$

$$\int_{C_r} \to 0 \text{ as } r \to 0 \text{ and } \int_{CR} \to 0 \text{ as } R \to \infty$$

$$\left(1-3^{2\pi\frac{i}{3}}\right)\int_0^\infty \frac{x^{\dot{\bar{3}}}}{(x+1)^2}dx = 2\pi i \,\mathrm{Res}\left(f(z),-1\right)$$

$$\mathrm{Res}\left(f(z),-1\right) = \underset{z\to -1}{\mathrm{limit}}\frac{d}{dz}z^{\frac{1}{2}} = \underset{z\to -1}{\mathrm{limit}} \qquad \frac{1}{3}z^{-\frac{2}{3}} = \frac{1}{3}(-1)^{-\frac{2}{3}}$$

$$= \frac{e^{-2\pi\frac{i}{3}}}{3}$$

$$\Rightarrow \int_0^\infty \frac{x^{\frac{1}{3}}}{(x+1)^2}dx = 2\pi i\frac{e^{-2\pi\frac{i}{3}}}{3\left(1-e^{2\pi\frac{i}{3}}\right)} = \frac{2\pi}{3\sqrt{3}}$$

4.

$$G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin \pi t}{1-t^2} e^{-i\omega t} dt = I_1 + I_2$$

where $I_1 = \text{ p.v. } \dfrac{-1}{4\pi i}\displaystyle\int_{-\infty}^{\infty} \frac{e^{it(\pi-\omega)}}{(t-1)(t+1)} dt$

and $I_2 = \text{ p.v. } \dfrac{1}{4\pi i}\displaystyle\int_{-\infty}^{\infty} \frac{e^{-it(\pi+\omega)}}{(t-1)(t+1)} dt$

For $\omega \le \pi$, use a contour in the upper half-plane to get

$$I_1 = \frac{-1}{4\pi i}(\pi i)\,[\text{Res}\,(t=-1) + \text{Res}(t=1)]$$

$$= \frac{-1}{4}\left[\frac{-e^{i(\omega-\pi)}}{2} + \frac{e^{i(\pi-\omega)}}{2}\right] = \frac{-i}{4}\sin(\pi-\omega) = \frac{-i}{4}\sin\omega$$

For $\omega \ge \pi$, use a contour in the lower half-plane to get

$$I_1 = \frac{-1}{4\pi i}(-\pi i)\,[\text{Res}\,(t=-1) + \text{Res}\,(t=1)] = \frac{i}{4}\sin\omega$$

For $\omega \ge \pi$, use a contour in the lower half-plane to get

$$I_1 = \frac{-1}{4\pi i}(-\pi i)\,[\text{Res}\,(t=-1) + \text{Res}\,(t=1)] = \frac{i}{4}\sin\omega$$

For $\omega \le -\pi$, use a contour in the upper half-plane to get

$$I_2 = \frac{1}{4\pi i}(\pi i)\,[\text{Res}\,(t=-1) + \text{Res}\,(t=1)]$$

$$= \frac{1}{4}\left[\frac{-e^{i(\pi+\omega)}}{2} + \frac{e^{-i(\pi+\omega)}}{2}\right] = \frac{-i}{4}\sin(\pi+\omega) = \frac{i}{4}\sin\omega$$

For $\omega \ge -\pi$, use a contour in the lower half-plane to get

$$I_2 = \frac{1}{4\pi i}(-\pi i)\,[\text{Res}\,(t=-1) + \text{Res}\,(t=1)] = \frac{-i}{4}\sin\omega$$

Summarizing,

$$G(\omega) = I_1 + I_2 = \begin{cases} \dfrac{-i}{2}\sin\omega & \text{for } |\omega| \le \pi \\ 0 & \text{for } |\omega| \ge \pi \end{cases}$$