CO2013: Complex Analysis, Unit-1 Testbank

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<Unit 1-1: Complex Number>

1. <2016u1-1> Use the De Moivre's formula to establish the identity (you NEED TO give complete derivation to the result!),

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta=\frac{1}{2}+\frac{\sin[(n+\frac{1}{2})\theta]}{2\sin(\theta/2)}.$$

2. <2018u1-1> (a) Find all the values of $\left(\frac{2i}{1+i}\right)^{1/6}$.
(b) Solve the equation $z^2-2z+i=0$.

3. <2019u1-1> Prove that $|z_1-z_2|^2=|z_1|^2-2\text{Re}(z_1\bar{z}_2)+|z_2|^2$.

4. <2019u1-1>
(a) Using the De Moivre's formula to establish the identities for $\cos^4\theta$ in terms of $\cos 2\theta$ and $\cos 4\theta$.
(b) Compute the integral

$$\int_0^{2\pi}\cos^4\theta d\theta.$$

5. <2019u1-1> Find all the values of the following:
(a) $(-16)^{1/4}$, (b) $(\frac{2i}{1+i})^{1/6}$.

6. <2019u1-1> Sove the equations

$$\frac{z^2-3z+1}{3-2z}=i$$

7. <2020u1-1> Prove that $\left(\frac{1+i\tan\theta}{1-i\tan\theta}\right)^n=\frac{1+i\tan n\theta}{1-i\tan n\theta}$, where n is any integer.

8. <2020u1-1> Find all solutions of the equation $z^6+z^3+1=0$ in the form of $x+iy$. ($\sin 20^o=0.342$, $\sin 40^o=0.643$, $\sin 80^o=0.985$, $\cos 20^o=0.94$, $\cos 40^o=0.766$, $\cos 80^o=0.174$)

<Unit 1-2: Complex Analytic Functions>

1. <2016u1-2> Prove that $f(z) = \bar{z}$ is continuous on the whole plane but is nowhere differentiable.

2. <2016u1-2, 2019u1-2> Let $z = x + iy$, where $x, y \in \mathbb{R}$, show that
$$f(z) = e^{x^2-y^2}[\cos(2xy) + i\sin(2xy)]$$
is entire, and find its derivative.

3. <2016u1-2> Find a harmonic conjugate of $u(x, y)$ with $u(x, y) = \sin x \cosh y$.

4. <2018u1-2> Show that $f(z) = |z|^2$ is differentiable at $z = 0$ but is not differentiable at any other point.

5. <2018u1-2> If u and v are expressed in terms of polar coordinates (r, θ), show that the Cauchy-Riemann equations can be written in the form
$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

6. <2018u1-2> Find a harmonic conjugate of $u = \ln|z|$ for $\text{Re}\{z\} > 0$.

7. <2018u1-2> Suppose $f(z) = u(r, \theta) + iv(r, \theta)$ is analytic in a domain D not containing the origin. The Cauchy-Riemann equations are in the form $ru_r = v_\theta$ and $rv_r = -u_\theta$.
   (a) Show that $u(r, \theta)$ satisfies Laplace's equation in polar coordinates:
$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$
   (b) Let $u(r, \theta) = r^3\cos 3\theta$. Find a harmonic conjugate $v(r, \theta)$ of $u(r, \theta)$.

8. <2019u1-2> Let
$$f(z) = \frac{x^2 + iy^2}{|z|^2}$$
when $z \neq 0$, and let $f(0) = 1$. Show that $f(z)$ is not continuous at $z_0 = 0$

9. <2019u1-2> Construct an analytic function whose real part is $u(x, y) = x^3 - 3xy^2 + y$.

10. <2020u1-2> Show that $f(z) = |z|$ is nowhere differentiable.

11. <2020u1-2> Show that the function $f(z) = (x^2 + y) + i(y^2 - x)$ is nowhere analytic.

12. <2020u1-2> Show that if v is a harmonic conjugate of u in a domain D, then uv is harmonic in D.

13. <2020u1-2> Construct an analytic function $f(z)$ in terms of z, whose real part is

$$u(x, y) = \frac{y}{x^2 + y^2},$$

where $z = x + iy$.

14. <2020u1-2> Suppose that $u(r, \theta) = r^2 \sin(2\theta)$, construct an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ in terms of z, where $z = re^{i\theta}$.

15. <2020u1-2> (a) Verify that $u(x, y) = x^3 - 3xy^2 - 5y$ is harmonic in the entire complex plane. (b) Find a harmonic conjugate of $u(x, y)$.

<Unit 1-3: Complex Elementary Functions>

1. <2016u1-3> Verify the identity

$$\sin z_2 - \sin z_1 = 2\cos\left(\frac{z_2+z_1}{2}\right)\sin\left(\frac{z_2-z_1}{2}\right),$$

and find the relationship between z_1 and z_2 if $\sin z_1 = \sin z_2$.

2. <2016u1-3> Solve the following equations:
(a) $\mathrm{Log}(z^2-1) = \frac{i\pi}{2}$. (b) $e^{2z} + e^z + 1 = 0$.

3. <2016u1-3> Find the following values:
(a) $(1+i)^{1-i}$. (b) the principal value of $(1+i)^{1+i}$.

4. <2018u1-3> Find z such that $e^z = (1+i)/\sqrt{2}$.

5. <2018u1-3> Solve the following equations:
(a) $e^z = 2i$. (b) $e^{2z} + e^z + 1 = 0$.

6. <2018u1-3> (a) Show that $\sin^{-1} z = -i\log[iz + (1-z^2)^{1/2}]$.
(b) Find the solutions of the equation $\sin z = 2$.

7. <2018u1-3> Show that $\sinh^{-1} z = \log\left[z+(z^2+1)^{1/2}\right]$ and $\tanh^{-1} z = \dfrac{1}{2}\log\left(\dfrac{1+z}{1-z}\right)$, $z \neq \pm 1$.

8. <2019u1-3> (a) Show that $\cos^{-1} z = -i\log\left[z + (z^2-1)^{1/2}\right]$.
(b) Find $\dfrac{d}{dz}\cos^{-1} z$.
(c) Find the solutions of the equation $\cos z = 2i$.

9. <2020u1-3> Solve the equation $\mathrm{Log}(z^2-1) = \frac{\pi i}{2}$ with the solution in the format of $x+iy$, where x and y are real values.

10. <2020u1-3> Express the following complex numbers in the format of $x+iy$, where x and y are real values:
(a) $2^{\pi i}$ (b) $(1-i)^{1+i}$ (c) $(-1)^{2/3}$

11. <2020u1-3> By using the formula $e^{iz} = \cos z + i\sin z$, solve the equation

$$\sin z = 2$$

for z in terms of natural logarithm.