Unit-1
Complex Analytic Functions

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Unit 1-1
Complex Numbers

Definition of Complex Numbers

Definition 1.1 *A complex number z is an ordered pair (x, y) of real numbers x and y, written as*

$$z = f(x, y)$$

where x is called the real part and y the imaginary part of z, written as

$$x = Re\{z\}, \quad y = Im\{z\}$$

Notes:

1. Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. Example:

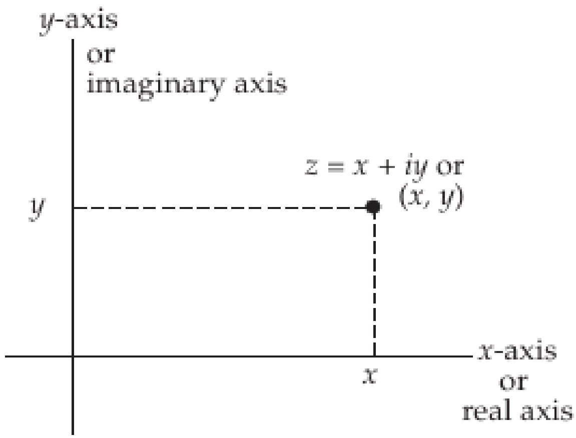
$$(2, 3) \neq (3, 2)$$

2. $(0, 1)$ is called the imaginary unit and is denoted by i, $i = (0, 1)$.

3. Definition of $i = \sqrt{-1}$ and $i^2 = -1$.

Definition of Complex Plane

Definition 1.2 z-plane: *We can represent the number $z = x + iy$ by a position vector in the xy plane. When the xy plane is used for displaying complex numbers, it is called the complex plane, or the z plane.*



Operation of Complex Numbers

Let $z = x + iy$.

- Addition

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

- Subtraction

$$z_1 - z_2 = (x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$$

- Multiplication

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (\_\_\_\_\_\_\_\_\_\_, \_\_\_\_\_\_\_\_\_\_)$$

- Division

$$\frac{z_1}{z_2} = \frac{(x_1, y_1)}{(x_2, y_2)} = \left(\frac{\ }{\_\_\_\_\_\_\_\_\_\_}, \frac{\ }{\_\_\_\_\_\_\_\_\_\_}\right), \quad \text{for } z_2 \neq 0.$$

Operation of Complex Numbers

The familiar commutative, associative, and distributive laws hold for complex numbers:

- Commutative laws:

$$z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1$$

- Associative laws:

$$z_1 + (z_2 + z_2) = (z_1 + z_2) + z_3, \quad z_1(z_2 z_3) = (z_1 z_2) z_3$$

- Distributive laws:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

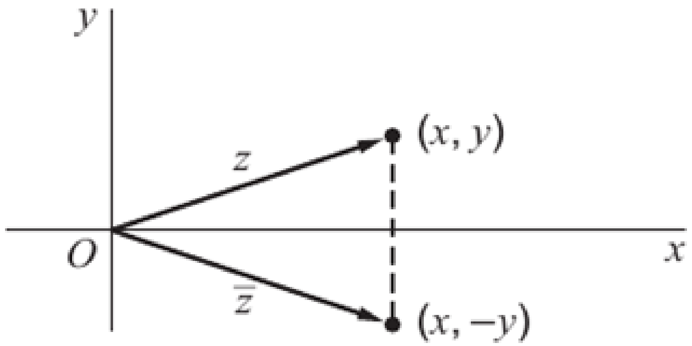
Complex Conjugate

Definition 1.3 *The conjugate of* z*, denoted* $\bar{z}$*, is the complex number* $(x, -y) = x - iy$.

Note: $z = x + iy, \bar{z} = x - iy,$
$\Longrightarrow \; x = \frac{1}{2}(z + \bar{z}), y = \frac{1}{2}(z - \bar{z}), Re\{iz\} = -Im\{z\}, Im\{iz\} = Re\{z\}.$

Properties:

- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$



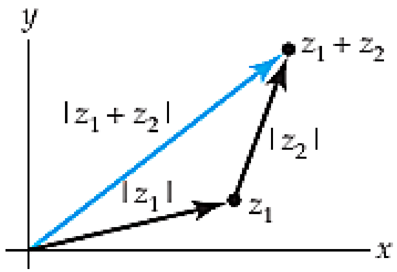
Modulus

Definition 1.4 *Modulus of z*, $|z| = \sqrt{x^2 + y^2}$.

Note: $|z|^2 = (Re\{z\})^2 + (Im\{z\})^2$.

Properties:

- $|z|^2 = z\bar{z}$ and $|z| = \sqrt{z\bar{z}}$
- $|z_1 z_2| = |z_1||z_2|$ and $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $|z^2| = |z|^2$



Triangle Inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$

Generalized triangle inequality:

$$\left|\sum_{k=1}^{N} z_k\right| \leq \sum_{k=1}^{N} |z_k|$$

Example 1.5 $z_1 = 1 + i, z_2 = -2 + 3i$
$|z_1 + z_2| = |-1 + 4i| = \sqrt{17} \doteq 4.123 < \sqrt{2} + \sqrt{13} \doteq 5.020$

Polar Form

Definition 1.6 *In addition to the xy-coordinate in the complex plane, we also employ the polar coordinates* r, θ *defined by*

$$x = r\cos\theta \quad \textit{and} \quad y = r\sin\theta$$

the polar form of $z = x + iy$ *at the point* P

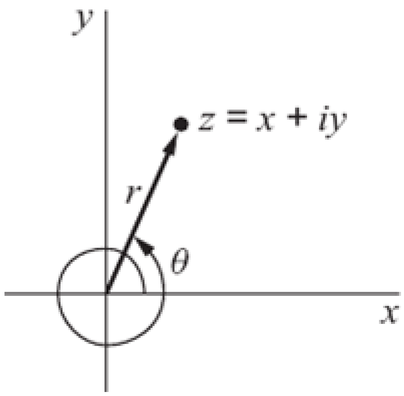
$$z = r(\cos\theta + i\sin\theta) \triangleq r\mathrm{cis}\theta$$

where r *is the absolute value of* z, $r = |z|$,

$$r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

and θ *is the argument of* z, $\theta = \arg z$

$$tan\theta = \frac{y}{x}$$



θ *is the directed angle from the positive x-axis to* $\overline{OP}$ *(counterclockwise).*

Note: $\theta = \alpha + 2n\pi, -\pi < \alpha \leq \pi, n = \pm 1, \pm 2, ...$ (integer multiples of 2π)

Principal Argument

Definition 1.7 *The principal argument of* z *is defined as* $\operatorname{Arg} z$, *where* $-\pi < \alpha = \operatorname{Arg} z \leq \pi$.

Note: $\arg z = \operatorname{Arg} z + 2n\pi, n = 0, \pm 1, \pm 2, \ldots.$

EXAMPLE 1.7 The complex number $-1 - i$, which lies in the third quadrant, has principal argument $-3\pi/4$. That is,

$$\operatorname{Arg}(-1 - i) = -3\pi/4$$

Multiplication and Division of Polar Form

Suppose

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

where θ_1 and θ_2 are any arguments of z_1 and z_2, respectively. Then

$$z_1 z_2 = r_1 r_2 \left[\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i\left(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2\right)\right]$$

and, for $z_2 \neq 0$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 + i\left(\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2\right)\right].$$

With polar form,

$$z_1 z_2 = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right].$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \quad \text{and}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$$

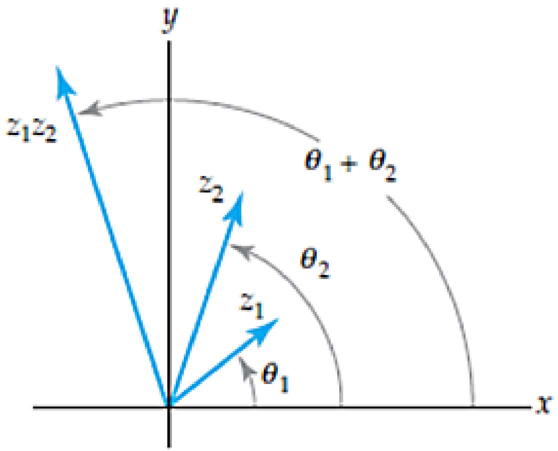


Figure 1.11 $\arg(z_1 z_2) = \theta_1 + \theta_2$

‡ $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ and $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

Notation of Euler's Equation

- Euler's equation enables us to write the polar form of a complex number as

$$z = r\,\mathrm{cis}\,\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

- We can drop the awkward "cis" artifice and use, as the standard polar representation,

$$z = re^{i\theta} = |z|\,e^{i\arg z}$$

Note: More generally, the circle $|z - z_0| = R$, whose center is z_0 and whose radius is R, has the parametric representation

$$z = z_0 + Re^{i\theta} \qquad 0 \le \theta \le 2\pi$$

Example 1.5.1 The equation $|z - 1 + 3i| = 2$ represents the circle whose center is $z_0 = (1, -3)$ and whose radius is $R = 2$.

Complex Exponential Functions

- Property

$$e^0 = 1,$$

$$e^{z_1} e^{z_2} = e^{z_1 + z_2},$$

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2},$$

$$(e^{z_1})^n = e^{n z_1} \text{ for } n = 0, \pm 1, \pm 2, \ldots .$$

- Let $z = x + iy$

$$e^z = e^{x+iy} = e^x e^{iy}$$

- Differentiation

$$\frac{de^z}{dz} = e^z$$

- *The complex exponential function is periodic.*

$$e^{z+2\pi i} = e^z$$

Proof of Euler's Equation

Theorem

If $z = x + iy$, then e^z is defined to be the complex number

$$e^z = e^x(\cos y + i\sin y).$$

Proof:

Example: Show that Euler's equation is formally consistent with the complex Taylor series expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

Solution

Property of Euler's Equation

Property

$$\cos\theta = \mathrm{Re}\left\{\frac{e^{i\theta}+e^{-i\theta}}{2}\right\} \qquad \sin\theta = \mathrm{Im}\left\{\frac{e^{i\theta}-e^{-i\theta}}{2i}\right\}$$

$$z_1 z_2 = r_1 e^{i\theta_1}\cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \qquad \overline{z} = re^{-i\theta}$$

Example: Show that $\cos(15^o) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ and $\sin(15^o) = \dfrac{\sqrt{6}-\sqrt{2}}{4}$.

Sol:

De Moivre's Formula

Note: Integer powers of z

$$z^n = r^n(\cos n\theta + i \sin n\theta).$$

When $n = 0$, we get the familiar result $z^0 = 1$.

Example: Compute z^3 for $z = -\sqrt{3} - i$.

Sol:

Multiple-Angle Formula

By De Moivre's Formula:

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$
$$= (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta).$$

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$
$$= (\cos\theta + i\sin\theta)\left[(\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta)\right]$$
$$= \cos^3\theta - \cos\theta\sin^2\theta - 2\cos\theta\sin^2\theta$$
$$+ i(\cos^2\theta\sin\theta - \sin^3\theta + 2\cos^2\theta\sin\theta)$$

$$\Rightarrow \begin{cases} \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \\ \qquad = \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\ \qquad = 4\cos^3\theta - 3\cos\theta. \\ \sin 3\theta = -\sin^3\theta + 3\sin\theta\cos^2\theta \\ \qquad = -\sin^3\theta + 3\sin\theta(1-\sin^2\theta) \\ \qquad = 3\sin\theta - 4\sin^3\theta. \end{cases}$$

Extension:

$$\begin{cases} \cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta. \\ \sin 5\theta = \sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta. \end{cases}$$

Example: Compute the integral

$$\int_0^{2\pi} \cos^4 \theta \, d\theta$$

Solution

Integer Powers of a Complex Number

Example 1: Compute $(1+i)^{20}$

Example 2: Compute $(-\sqrt{3}-i)^{30}$

Example 3: Compute $(a)\int_0^{2\pi}\cos^8\theta d\theta \qquad (b)\int_0^{2\pi}\sin^6(2\theta)d\theta$

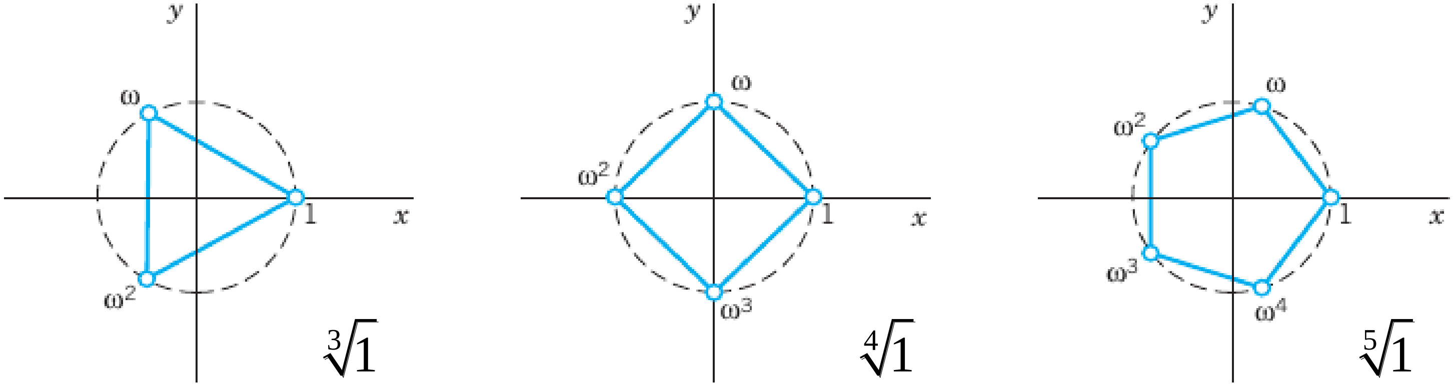
Sol:

Fractional Powers of a Complex Number - *N*th Roots of Unity

$$\text{Consider } z = 1, \text{ then } |z| = r = 1$$

$$1^{\frac{1}{n}} = \sqrt[n]{1} \triangleq \omega^k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k = 0, 1, \cdots, n-1$$

These n values are called the nth roots of unity.



Example: $\sqrt[8]{1} = ?$

Solution:

Root of a Complex Number

Roots	If $z = w^n$ $(n=1,2,\cdots)$, then to each value of w there corresponds one value of z. Each of these values is called an nth root of z.
	$w = \sqrt[n]{z}$
	Note: w is multi-valued. (n-valued)
Polar form	Let $z = r(\cos\theta + i\sin\theta)$
	$w = R(\cos\phi + i\sin\phi)$
	Then
	$w^n = R^n(\cos n\phi + i\sin n\phi) = z$
	$n\phi = \theta + 2k\pi,\quad k=0,1,2,\cdots,n-1$
	$\Rightarrow \phi = \frac{\theta}{n} + \frac{2k\pi}{n}$
	$\therefore \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$

Remark:

$$\text{If } z = r\,\text{cis}\,\theta, \quad \theta = \text{Arg } z, \; r \in R$$

$$z^{\frac{1}{n}} = \sqrt[n]{r}\,\text{cis}\left(\frac{\text{Arg } z + 2k\pi}{n}\right), \quad k = 0, 1, 2, \cdots, n-1$$

Example 1: Find all the cube roots of $8i$

Sol:

Example 2: Compute $\left(\dfrac{2i}{1+i}\right)^{\frac{1}{6}}$

Sol:

Solving Complex Coefficients Equations

Example 1: Solve $x^4 + 324 = 0$

Solution:

Ex1: solve $z^2 - (6-2i)z + 17 - 6i = 0$

Sol:

Ex2: solve $z^2 + 2z + (1-i) = 0$

Sol:

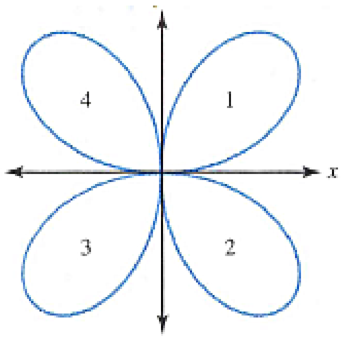
Topology of Complex Numbers

Define a curve to be the range of a continuous complex-valued function $z(t)$ defined on the interval [a, b]. That is, a curve C is the range of a function given by z(t) = (x(t) , y(t)) = x(t) + iy(t) , for $a \leq t \leq b$， where both x(t) and y(t) are continuous real-valued functions. If both x(t) and y(t) are differentiable, we say that **the curve is smooth**.

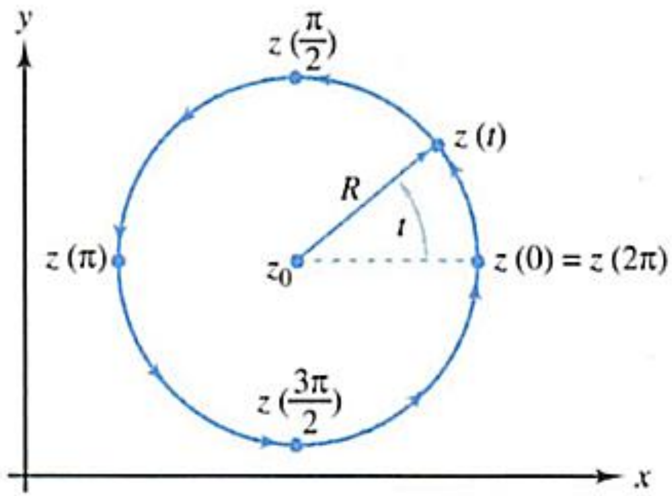
Note that, with this parametrization, we are specifying a direction for the curve C, saying that C is a curve that goes from the **initial point** z(a) =(x(a), y(a))=x(a)+iy(a) to the **terminal point** z(b)= (x(b), y(b)) = x(b) + iy(b).

A curve C have the property that z(a) = z(b) is said to be a **closed curve**.

Example: The curve $x(t) = \sin 2t \cos t,\ y(t) = \sin 2t \sin t$ for $0 \leq t \leq 2\pi$,



Example: The simple closed curve $z(t) = z_0 + Re^{it}$, for $0 \leq t \leq 2\pi$.



We use the notation $C_R^+(z_0)$ to indicate that the parametrization we chose for this simple closed curve resulted in a positive orientation; $C_R^-(z_0)$ denotes the same circle, but with a negative orientation. (In both cases, *counterclockwise* denotes the positive direction.) Using notation that we have already introduced, we get $C_R^-(z_0) = -C_R^+(z_0)$.

Definition of Disk

Open disk of radius $\rho > 0$ about a is denoted as

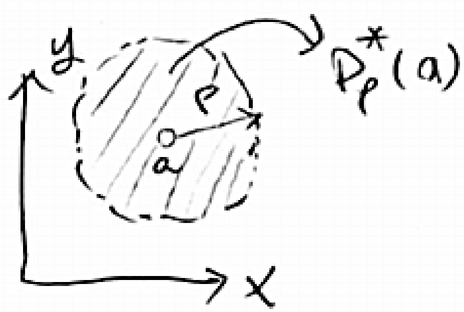
$$D_\rho(a) = \{z : |z - a| < \rho\}$$

Closed disk of radius ρ centered at a

$$\overline{D_\rho}(a) = \{z : |z - a| \le \rho\}$$

Punctured disk of radius ρ centered at a.

$$D_\rho^*(a) = \{z : 0 < |z - a| < \rho\}.$$

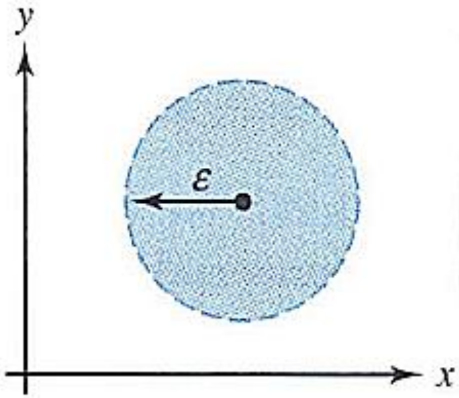


a is punctured

Neighborhood

Definition: An open circular disk centered at z_0 is also called a **neighborhood** of z_0, or, an **ε-neighborhood** of z_0.

It is the set of all points satisfying the inequality $\{z : |z - z_0| < \varepsilon\}$ and is denoted $D_\varepsilon(z_0)$.



$$D_\varepsilon(z_0) = \{z : |z - z_0| < \varepsilon\}$$

■ **EXAMPLE 1.23** The solution sets of the inequalities $|z| < 1$, $|z - i| < 2$, and $|z + 1 + 2i| < 3$ are neighborhoods of the points 0, i, and $-1 - 2i$, with radii 1, 2, and 3, respectively. They can also be expressed as $D_1(0)$, $D_2(i)$, and $D_3(-1 - 2i)$.

Deleted Neighborhood

Definition: Occasionally, we will need to use a neighborhood of z_0 that also excludes z_0. Such a neighborhood is defined by the simultaneous inequality $0 < |z - z_0| < \varepsilon$ and is called a **deleted neighborhood** of z_0 (punctured disk).

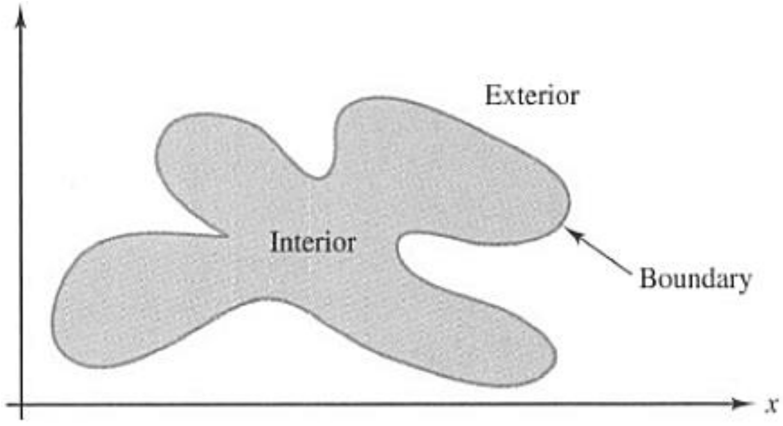
Example: $|z| < 1$ defines a neighborhood of the origin, whereas $0 < |z| < 1$ defines a deleted neighborhood of the origin.

Interior and Boundary Points

- The point z_0 is said to be an **interior point** of the set S provided that there exists an ε-neighborhood of z_0 that contains only points of S.

- z_0 is called an **exterior point** of the set S if there exists an ε-neighborhood of z_0 that contains no points of S.

- If z_0 is neither an interior point nor an exterior point of S , then it is called a **boundary point** of S and has the property that each ε-neighborhood of z_0 contains both points in S and points not in S.

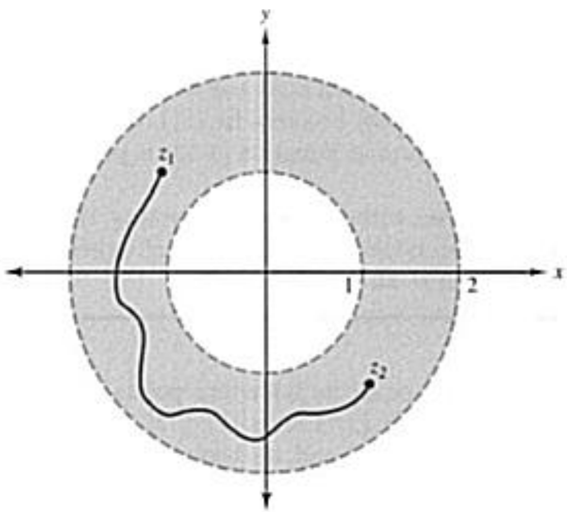


Open Set, Closed Set, and Connected Set

- **Open Set**: A set S is called an open set if every point of S is an interior point of S.

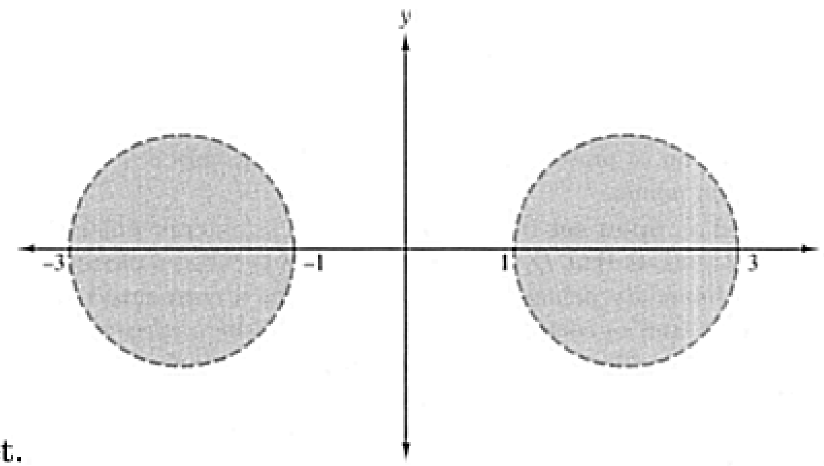
- **Closed Set**: A set S is called a closed set if it contains all its boundary points.

- **Connected Set**: A set S is said to be a connected set if every pair of points z_1 and z_2 contained in S can be joined by a curve that lies entirely in S.



The annulus $A = \{z : 1 < |z| < 2\}$ is a connected set.

The set $B = \{z : |z+2| < 1 \text{ or } |z-2| < 1\}$ is not a connected set.



Domain and Region

- We call a connected open set a **domain**.

  Ex: The open unit disk $D_1(0)$ = {z : |z| < 1} is a domain.
  The closed unit disk $\bar{D}_1(0)$ = {z : |z| $\leq$ 1} is not a domain.

- A domain, together with some, none, or all its boundary points, is called a **region**. A set formed by taking the union of a domain and its boundary is called a **closed region.**

  Ex: The horizontal strip {z : 1 < Im (z) $\leq$ 2} is a region.

- A set S is said to be a **bounded set** if it can be completely contained in some closed disk. A set that cannot be enclosed by any closed disk is called an **unbounded set**.

  Ex: The rectangle given by {z : |x| <4 and |y|< 3} is bounded because it is contained inside the disk $\bar{D}_5(0)$.