Unit 1-2
Complex Analytic Functions

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Complex-Valued Functions

A function f defined on D is a rule that assigns to every z in D a complex number w, called the value of f at z,

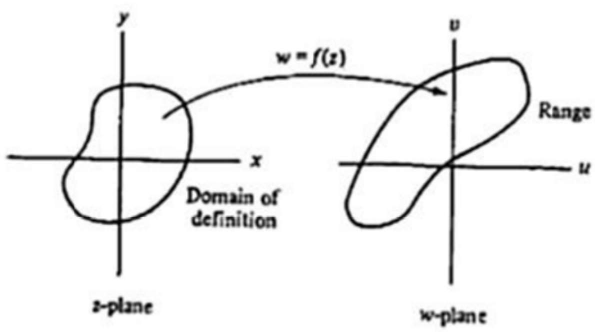
$$w = f(z)$$

where

$z \in D$: complex variable

D: the domain of f. (In most cases, D will be open and connected)

The set of all values of a function f is called the range of f.



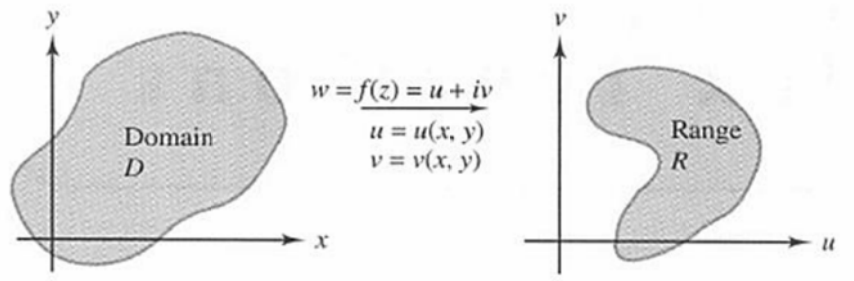
Complex Function Expression

Just as z can be expressed by its real and imaginary parts, $z = x + iy$, we write $f(z) = w = u + iv$, where u and v are the real and imaginary parts of w, respectively. Doing so gives us the representation

$$w = f(z) = f(x, y) = f(x + iy) = u + iv.$$

Because u and v depend on x and y, they can be considered to be real-valued functions of the real variables x and y; that is,

$$u = u(x, y) \quad \text{and} \quad v = v(x, y).$$



EXAMPLE 2.1 Write $f(z) = z^4$ in the form $f(z) = u(x, y) + iv(x, y)$.

EXAMPLE 2.2 Express the function $f(z) = \overline{z}\,\mathrm{Re}(z) + z^2 + \mathrm{Im}(z)$ in the form $f(z) = u(x, y) + iv(x, y)$.

EXAMPLE 2.3 Express $f(z) = 4x^2 + i4y^2$ by a formula involving the variables z and $\overline{z}$.

Polar Expression

If the polar coordinates r and θ, instead of x and y, are used, then

$$u + iv = f(re^{i\theta})$$

where $w = u + iv$ and $z = re^{i\theta}$. In that case, we may write

$$f(z) = u(r, \theta) + iv(r, \theta).$$

Ex 1: Write the function

$$f(z) = z + \frac{1}{z} \qquad (z \neq 0)$$

in the form $f(z) = u(r, \theta) + iv(r, \theta)$.

Ans.

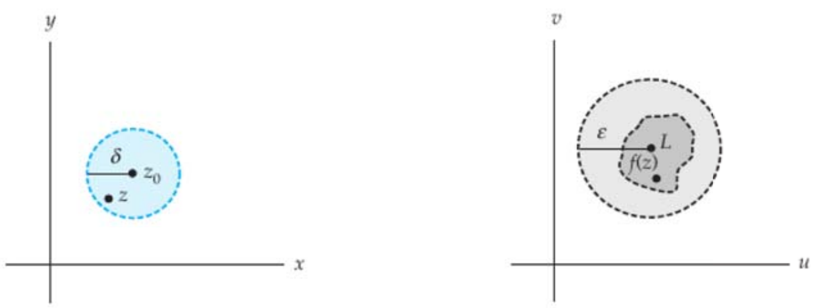
Ex 2: Express $f(z) = z^5 + 4z^2 - 6$ in polar form.

Sol:

Limit of a Complex Function

Definition

Suppose that a complex function f is defined in a deleted neighborhood of z_0 and suppose that L is a complex number. The **limit of f as z tends to z_0 exists and is equal to L**, written as $\lim_{z \to z_0} f(z) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.



(a) Deleted δ-neighborhood of z_0 **(b)** ε-neighborhood of L

Epsilon-Delta Proof of a Limit

Example: show that if $f(z) = i\bar{z}/2$ in the open disk $|z| < 1$, then

$$\lim_{z \to 1} f(z) = \frac{i}{2}$$

Proof:

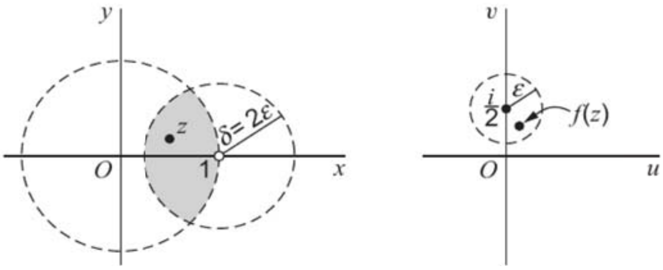
the point 1 being on the boundary of the domain of definition of f. Observe that when z is in the disk $|z| < 1$,

$$\left| f(z) - \frac{i}{2} \right| = \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| = \frac{|z-1|}{2}.$$

Hence, for any such z and each positive number ε

$$\left| f(z) - \frac{i}{2} \right| < \varepsilon \quad \text{whenever} \quad 0 < |z-1| < 2\varepsilon.$$

Thus condition (2) is satisfied by points in the region $|z| < 1$ when δ is equal to 2ε or any smaller positive number.



Example: Prove that $\lim_{z \to i} z^2 = -1$

Proof:

Solution. We must show that for given $\varepsilon > 0$ there is a positive number δ such that

$$|z^2 - (-1)| < \varepsilon \quad \text{whenever} \quad 0 < |z - i| < \delta.$$

So we express $|z^2 - (-1)|$ in terms of $|z - i|$:

$$z^2 - (-1) = z^2 + 1 = (z - i)(z + i) = (z - i)(z - i + 2i).$$

It follows from the properties of $|a+b| \le |a| + |b|$

$$\left| z^2 - (-1) \right| = |z - i||z - i + 2i| \le |z - i|(|z - i| + 2). \quad (1)$$

Now if $|z - i| < \delta$ the right-hand member of (1) is less than $\delta(\delta + 2)$; so to ensure that it is less than ε, we choose δ to be smaller than each either of the numbers $\varepsilon/3$ and 1:

$$|z - i|(|z - i| + 2) < \frac{\varepsilon}{3}(1 + 2) = \varepsilon. \quad \blacksquare$$

Criterion for the Nonexistence of a Limit

For limits of complex functions, z is allowed to approach z_0 from *any* direction in the complex plane, that is, along any curve or path through z_0. See Figure 2.53. In order that $\lim_{z \to z_0} f(z)$ exists and equals L, we require that $f(z)$ approach the same complex number L along every possible curve through z_0. Put in a negative way:

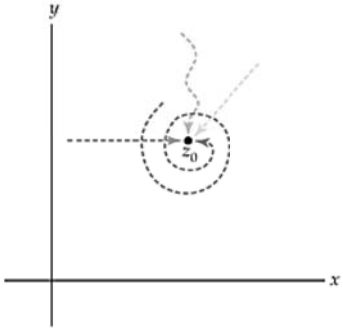


Figure 2.53 Different ways to approach z_0 in a limit

Criterion for the Nonexistence of a Limit

If f approaches two complex numbers $L_1 \neq L_2$ for two different curves or paths through z_0, then $\lim_{z \to z_0} f(z)$ does not exist.

EXAMPLE 2.16 Show that the function $u(x, y) = \frac{xy}{x^2+y^2}$ does not have a limit as (x, y) approaches $(0, 0)$.

Ex: Show that $\lim_{z \to 0} \frac{z}{\bar{z}}$ does not exist.

Theorem of Limits

Suppose that $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$, and $L = u_0 + iv_0$. Then $\lim_{z \to z_0} f(z) = L$ if and only if

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0.$$

■ **EXAMPLE 2.18** Show that $\lim_{z \to 1+i} \left(z^2 - 2z + 1\right) = -1$.

Theorem of Limits

Suppose that f and g are complex functions. If $\lim_{z \to z_0} f(z) = L$ and $\lim_{z \to z_0} g(z) = M$, then

(i) $\lim_{z \to z_0} cf(z) = cL$, c a complex constant,

(ii) $\lim_{z \to z_0} (f(z) \pm g(z)) = L \pm M$,

(iii) $\lim_{z \to z_0} f(z) \cdot g(z) = L \cdot M$, and

(iv) $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{L}{M}$, provided $M \neq 0$.

Example: Compute the limits

(a) $\lim_{z \to i} \dfrac{(3+i)z^4 - z^2 + 2z}{z+1}$

(b) $\lim_{z \to 1+\sqrt{3}i} \dfrac{z^2 - 2z + 4}{z - 1 - \sqrt{3}i}$

Sol:

(a)

(b)

Continuity of a Complex Function

Definition A complex function f is **continuous at a point z_0** if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Criteria for Continuity at a Point

A complex function f is continuous at a point z_0 if each of the following three conditions hold:

(i) $\lim_{z \to z_0} f(z)$ *exists,*

(ii) f *is defined at* z_0*, and*

(iii) $\lim_{z \to z_0} f(z) = f(z_0)$.

Remark: A function whose limit exists at a certain point does not imply that it is continuous at that point.

Example:

even though $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$

$f(x) = \frac{x^2 - 1}{x - 1}$ is not continuous at $x = 1$ because $f(1)$ is not defined.

Real and Imaginary Parts of a Continuous Function

Suppose that $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$. Then the complex function f is continuous at the point z_0 if and only if both real functions u and v are continuous at the point (x_0, y_0).

EXAMPLE 7

Show that the function $f(z) = \bar{z}$ is continuous on $\mathbf{C}$.

Properties of Continuous Functions

If f and g are continuous at the point z_0, then the following functions are continuous at the point z_0:

(i) cf, c a complex constant,

(ii) $f \pm g$,

(iii) $f \cdot g$, and

(iv) $\dfrac{f}{g}$ provided $g(z_0) \neq 0$.

Remark: Polynomial functions are continuous on the entire complex plane $\mathbf{C}$.

Derivatives

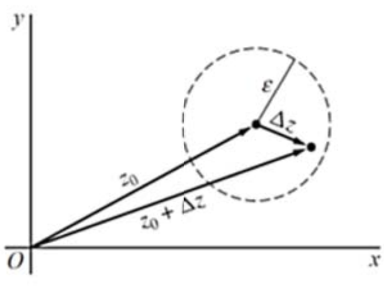
Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \varepsilon$ of a point z_0. The *derivative* of f at z_0 is the limit

(1) $$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

and the function f is said to be *differentiable* at z_0 when $f'(z_0)$ exists.

one can write that definition as

(2) $$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$



When taking form (2) of the definition of derivative, we often drop the subscript on z_0 and introduce the number

$$\Delta w = f(z + \Delta z) - f(z),$$

which denotes the change in the value $w = f(z)$ of f corresponding to a change Δz in the point at which f is evaluated. Then, if we write dw/dz for $f'(z)$, equation (2) becomes

(3) $$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}.$$

Ex: Show that $f(z) = \overline{z}$ is not differentiable at any point $\mathbf{z}$.

Sol:

EXAMPLE 3 A Function That Is Nowhere Differentiable

Show that the function $f(z) = x + 4iy$ is not differentiable at any point z.

Differentiability Implies Continuity

If f is differentiable at a point z_0 in a domain D, then f is continuous at z_0.

Proof:

Differential Rules

Differentiation Rules

Constant Rules: $\frac{d}{dz}c = 0$ and $\frac{d}{dz}cf(z) = cf'(z)$

Sum Rule: $\frac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z)$

Product Rule: $\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$

Quotient Rule: $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$

Chain Rule: $\frac{d}{dz}f(g(z)) = f'(g(z))\,g'(z).$

EXAMPLE 2 Using the Rules of Differentiation

Differentiate:

(a) $f(z) = 3z^4 - 5z^3 + 2z$ **(b)** $f(z) = \frac{z^2}{4z+1}$ **(c)** $f(z) = \left(iz^2 + 3z\right)^5$

Solution

(a) $12z^3 - 15z^2 + 2.$ **(b)** $\frac{4z^2 + 2z}{(4z+1)^2}$ **(c)** $5(iz^2 + 3z)^4(2iz + 3)$

L'Hopital's Rule

Suppose f and g are functions that are analytic at a point z_0 and $f(z_0) = 0,\ g(z_0) = 0$, but $g'(z_0) \neq 0$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

EXAMPLE 4 Using L'Hôpital's Rule

Compute $\lim_{z \to 2+i} \frac{z^2 - 4z + 5}{z^3 - z - 10i}$.

Analyticity

Definition

A complex function $w = f(z)$ is said to be **analytic at a point** z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

Remark:

A function $\boldsymbol{f}$ **is analytic in a domain** $\boldsymbol{D}$ if it is analytic at every point in D. The phrase "analytic *on* a domain D" is also used. Although we shall not use these terms in this text, a function f that is analytic throughout a domain D is called **holomorphic** or **regular**.

Entire Functions

Definition: A function that is analytic at every point z in the complex plane is said to be an **entire function**.

Theorem

(i) A polynomial function $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where n is a nonnegative integer, is an entire function.

(ii) A rational function $f(z) = \dfrac{p(z)}{q(z)}$, where p and q are polynomial functions, is analytic in any domain D that contains no point z_0 for which $q(z_0) = 0$.

Cauchy Riemann Equations

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exist and satisfy the **Cauchy-Riemann equations**

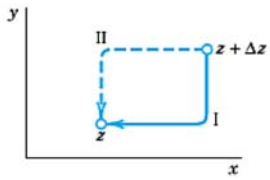
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Proof:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

We write $\Delta z = \Delta x + i\,\Delta y$. Then $z + \Delta z = x + \Delta x + i(y + \Delta y)$

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)] - [u(x, y) + iv(x, y)]}{\Delta x + i\,\Delta y}.$$



(a) We choose path I, $\Delta y = 0$ => $\Delta z = \Delta x$

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}.$$

$$f'(z) = u_x + iv_x.$$

(b) We choose path II, $\Delta x = 0$ => $\Delta z = i\Delta y$

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\,\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{i\,\Delta y}$$

$$f'(z) = -iu_y + v_y.$$

⇒ $u_x = v_y, \qquad u_y = -v_x$

Example 1:

The polynomial function $f(z) = z^2 + z$ is analytic for all z and can be written as $f(z) = x^2 - y^2 + x + i(2xy + y)$. Thus, $u(x, y) = x^2 - y^2 + x$ and $v(x, y) = 2xy + y$. For any point (x, y) in the complex plane we see that the Cauchy-Riemann equations are satisfied:

$$\frac{\partial u}{\partial x} = 2x + 1 = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

Example 2: Show that the function $f(z) = z^2 = x^2 - y^2 + i2xy$ is differentiable everywhere and find $f'(z)$.

Sol:

Non-analyticity

Criterion for Non-analyticity

If the Cauchy-Riemann equations are not satisfied at every point z in a domain D, then the function $f(z) = u(x, y) + iv(x, y)$ cannot be analytic in D.

EXAMPLE 2 Using the Cauchy-Riemann Equations

Show that the complex function $f(z) = 2x^2 + y + i(y^2 - x)$ is not analytic at any point.

Sufficient Condition for Analyticity

Suppose the real functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D. If u and v satisfy the Cauchy-Riemann equations at all points of D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

Remark:

A Sufficient Condition for Analyticity By themselves, the Cauchy-Riemann equations do not ensure analyticity of a function $f(z) = u(x, y) + iv(x, y)$ at a point $z = x + iy$. It is possible for the Cauchy-Riemann equations to be satisfied at z and yet $f(z)$ may not be differentiable at z, or $f(z)$ may be differentiable at z but nowhere else. In either case, f is not analytic at z.

Example: Show that $f(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$ is analytic except at z=0.

Sol:

(i) $u(x, y) = \frac{x}{x^2+y^2}$ and $v(x, y) = -\frac{y}{x^2+y^2}$ are continuous except at the point where $x^2+y^2 = 0$, that is, at $z = 0$.

(ii) the first four first-order partial derivatives

$$\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2},$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}, \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

are continuous except at $z = 0$.

(iii) $\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}$

the Cauchy-Riemann equations are satisfied except at $z = 0$.

Thus we conclude that f is analytic in any domain D that does not contain the point $z = 0$.

EX 1: Is the exponential function $f(z) = e^z$ analytic?

Sol:

EX 2: Show that the function $f(z) = x^3 + 3xy^2 + i(y^3 + 3x^2y)$ is differentiable on the the x- and y-axes but analytic nowhere.

Sol:

EX 3: Show that

$$f(z) = \begin{cases} \overline{z}^2/z & \text{when} \quad z \neq 0, \\ 0 & \text{when} \quad z = 0. \end{cases}$$

is not differentiable (analytic) at $z = 0$ even though the Cauchy-Riemann equations are satisfied at (0,0).

Solution:

Constant Analytic Functions

Suppose the function $f(z) = u(x,\ y) + iv(x,\ y)$ is analytic in a domain D.

(i) If $|f(z)|$ is constant in D, then so is $f(z)$.

(ii) If $f'(z) = 0$ in D, then $f(z) = c$ in D, where c is a constant.

Proof of (i):

Polar Form of Cauchy-Riemann Equations

Let $z = r(\cos\theta + i\sin\theta)$ and $f(z) = u(r,\theta) + iv(r,\theta)$. The Cauchy-Riemann equations are

$$u_r = \frac{1}{r} v_\theta \quad \text{and} \quad v_r = -\frac{1}{r} u_\theta$$

Proof:

Example: Let $f(z)=1/z$, find $f'(z)$.

Solution:

Laplace Equations

If $f(z) = u(x, y) + iv(x, y)$ *is analytic in a domain D, then both u and v satisfy* **Laplace's equation**

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

(∇^2 read "nabla squared") *and*

$$\nabla^2 v = v_{xx} + v_{yy} = 0,$$

in D and have continuous second partial derivatives in D.

Proof:

Harmonic Functions

Definition

A real-valued function ϕ of two real variables x and y that has continuous first and second-order partial derivatives in a domain D and satisfies Laplace's equation is said to be **harmonic** in D.

$$\nabla^2\phi = \phi_{xx}(x, y) + \phi_{yy}(x, y) = 0$$

Theorem

Suppose the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D. Then the functions $u(x, y)$ and $v(x, y)$ are harmonic in D.

EXAMPLE 1 **Harmonic Functions**

The function $f(z) = z^2 = x^2 - y^2 + 2xyi$ is entire. The functions $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ are necessarily harmonic in any domain D of the complex plane.

EXAMPLE 3. Since the function $f(z) = i/z^2$ is analytic whenever $z \neq 0$ and since

$$\frac{i}{z^2} = \frac{i}{z^2} \cdot \frac{\overline{z}^2}{\overline{z}^2} = \frac{i\overline{z}^2}{(z\overline{z})^2} = \frac{i\overline{z}^2}{|z|^4} = \frac{2xy + i(x^2 - y^2)}{(x^2 + y^2)^2},$$

the two functions

$$u(x, y) = \frac{2xy}{(x^2 + y^2)^2} \quad \text{and} \quad v(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

are harmonic throughout any domain in the xy plane that does not contain the origin.

Harmonic Conjugate

Definition

If we have a function $u(x, y)$ that is harmonic on the domain D and if we can find another harmonic function $v(x, y)$ such that the partial derivatives for u and v satisfy the Cauchy–Riemann equations throughout D, then we say that $v(x, y)$ is a **harmonic conjugate** of $u(x, y)$. It then follows that the function $f(z) = u(x, y) + iv(x, y)$ is analytic on D.

Example: Suppose that

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

Since these are the real and imaginary components, respectively, of the entire function $f(z) = z^2$, we know that v is a harmonic conjugate of u throughout the plane.

Question: Is u a harmonic conjugate of v?

Hint: check whether the function $g(z)=v+iu$ is analytic anywhere.

Finding Harmonic Conjugate for an Analytic Function

■ **EXAMPLE 3.13** Show that $u(x, y) = xy^3 - x^3y$ is a harmonic function, and find a conjugate harmonic function $v(x, y)$.

Solution:

Finding Harmonic Conjugate for an Analytic Function

Ex 1:

(a) Verify that the function $u(x,\ y) = x^3 - 3xy^2 - 5y$ is harmonic in the entire complex plane.

(b) Find the harmonic conjugate function of u.

Sol:

Ex 2: Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a harmonic conjugate function v of u. Also find $f(z)=u+iv$ in terms of z.

Sol: