Unit 1-3
Complex Elementary Functions

Prof. Dah-Chung Chang (張大中)
Department of Communication Eng.
National Central University
dcchang@ce.ncu.edu.tw

Polynomial and Rational Functions

polynomial functions of z are functions of the form

$$p_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

The degree is n if the complex constant a_n is nonzero.

rational functions are ratios of polynomials

$$R_{m,n}(z) = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m}{b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n}$$

The rational function has *numerator degree m* and *denominator degree n* , if $a_m \neq 0$ and $b_n \neq 0$.

The analyticity of these functions is quite transparent: polynomials are entire, and rational functions are analytic everywhere except for the zeros of their denominators.

Example 1

Carry out the deflation of the polynomial $z^3 + (2 - i)z^2 - 2iz$.

Solution.

Taylor Form of a Polynomial

The coefficients of $(z-z_0)^k$, in the expansion of a polynomial $p_n(z)$ in powers of $(z-z_0)$, is given by its *k*th derivative, evaluated at z_0, and divided by *k* factorial:

$$p_n(z)=\frac{p_n(z_0)}{0!}+\frac{p_n'(z_0)}{1!}(z-z_0)^1+\frac{p_n''(z_0)}{2!}(z-z_0)^2+\cdots+\frac{p_n^{(n)}(z_0)}{n!}(z-z_0)^n$$
$$=\sum_{k=0}^{n}\frac{p_n^{(k)}(z_0)}{k!}(z-z_0)^k$$

Ex: the Taylor form of $p_3(z)$ cenlered al its zero $z_0=2$ is

$$p_3(z)=12+10z-4z^2-2z^3$$
$$p_3(z)=(0)-\frac{30}{1!}(z-2)-\frac{32}{2!}(z-2)^2-\frac{12}{3!}(z-2)^3.$$

Notice: **Maclaurin form** is centered at 0 for the Taylor form; the standard form $p_3(z)$ is thus its Maclaurin form.

Poles and Zeros

The factored form:

$$R_{m,n}(z)=\frac{a_m(z-z_1)(z-z_2)\cdots(z-z_m)}{b_n(z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_n)}$$

where $\{z_k\}$ designates the zeros of the numerator and $\{\zeta_k\}$ designates those of the denominator. We assume that common zeros have been cancelled. The zeros of the numerator are zeros of $R_{m,n}(z)$; zeros of the denominator are called poles of $R_{m,n}(z)$.

Ex: Find all the poles and their multiplicities for

$$R(z)=\frac{(3z+3i)(z^2-4)}{(z-2)(z^2+1)^2}.$$

Sol:

We see that the only poles of $R(z)$ are at $z=i$ of multiplicity 2 and $z=-i$ of multiplicity 1.

Partial Fractional Decomposition

Theorem

$$R_{m,n}(z) = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m}{b_n (z-\zeta_1)^{d_1}(z-\zeta_2)^{d_2}\cdots(z-\zeta_r)^{d_r}}$$

is a rational function whose denominator degree $n = d_1 + d_2 + \cdots + d_r$ exceeds its numerator degree m, then $R_{m,n}(z)$ has a **partial fraction decomposition** of the form

$$\begin{aligned} R_{m,n}(z) = {} & \frac{A_0^{(1)}}{(z-\zeta_1)^{d_1}} + \frac{A_1^{(1)}}{(z-\zeta_1)^{d_1-1}} + \cdots + \frac{A_{d_1-1}^{(1)}}{(z-\zeta_1)} \\ & + \frac{A_0^{(2)}}{(z-\zeta_2)^{d_2}} + \cdots + \frac{A_{d_2-1}^{(2)}}{(z-\zeta_2)} \\ & + \cdots + \frac{A_0^{(r)}}{(z-\zeta_r)^{d_r}} + \cdots + \frac{A_{d_r-1}^{(r)}}{(z-\zeta_r)}, \end{aligned}$$

where the $\{A_s^{(j)}\}$ are constants. (The ζ_k's are assumed distinct.)

$$A_s^{(j)} = \lim_{z\to\zeta_j} \frac{1}{s!}\frac{d^s}{dz^s}\left[\left(z-\zeta_j\right)^{d_j} R_{m,n}(z)\right]$$

Ex 1: Reproduce the partial fraction decomposition of the rational function

$$R(z) = \frac{4z+4}{z(z-1)(z-2)^2}$$

Sol:

Ex 2: Reproduce the partial fraction decomposition of the rational function

$$R(z) = \frac{2z+1}{z(z-2)^2}$$

Sol:

Complex Exponential Functions

Definition The function e^z defined by

$$e^z = e^x \cos y + ie^x \sin y$$

is called the **complex exponential function.**

Theorem The exponential function e^z is entire and its derivative is given by:

$$\frac{d}{dz}e^z = e^z.$$

Proof:

EXAMPLE 1 **Derivatives of Exponential Functions**

Find the derivative of each of the following functions:

(a) $iz^4\left(z^2 - e^z\right)$ and (b) $e^{z^2-(1+i)z+3}$.

Solution

Example: Find number $z = x + iy$ such that $e^z = 1 + i$.

Solution:

Definition of Complex Trigonometric Functions

Definition

$$e^{iz} = \cos z + i \sin z$$

The complex **sine** and **cosine** functions are defined by:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \text{and} \quad \csc z = \frac{1}{\sin z}.$$

Periodicity

$$\sin\left(z + \frac{\pi}{2}\right) = \cos z, \quad \sin\left(z - \frac{\pi}{2}\right) = -\cos z,$$

$$\sin(z + 2\pi) = \sin z, \quad \sin(z + \pi) = -\sin z,$$

$$\cos(z + 2\pi) = \cos z, \quad \cos(z + \pi) = -\cos z.$$

Proof:

EXAMPLE 1 Values of Complex Trigonometric Functions

In each part, express the value of the given trigonometric function in the form $a + ib$.

(a) $\cos i$ **(b)** $\sin(2 + i)$ **(c)** $\tan(\pi - 2i)$

Solution:

Complex Trigonometric Identities

$$\sin(-z) = -\sin z \quad \cos(-z) = \cos z$$

$$\cos^2 z + \sin^2 z = 1$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin 2z = 2 \sin z \cos z, \quad \cos 2z = \cos^2 z - \sin^2 z,$$

Proof:

Derivatives of Complex Trigonometric Functions

$$\frac{d}{dz}\sin z = \cos z \qquad \frac{d}{dz}\cos z = -\sin z$$

$$\frac{d}{dz}\tan z = \sec^2 z \qquad \frac{d}{dz}\cot z = -\csc^2 z$$

$$\frac{d}{dz}\sec z = \sec z \tan z \qquad \frac{d}{dz}\csc z = -\csc z \cot z$$

Proof:

Complex Trigonometric Functions and Hyperbolic Functions

Definition: Hyperbolic Functions

$$\sinh y = \frac{e^y - e^{-y}}{2} \quad \text{and} \quad \cosh y = \frac{e^y + e^{-y}}{2}$$

$$\sin(iy) = i \sinh y \quad \text{and} \quad \cos(iy) = \cosh y.$$

$$\cosh^2 y = 1 + \sinh^2 y$$

$$\sin z = \sin x \cosh y + i \cos x \sinh y, \qquad |\sin z|^2 = \sin^2 x + \sinh^2 y,$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y, \qquad |\cos z|^2 = \cos^2 x + \sinh^2 y.$$

Proof

Complex Hyperbolic Functions and Their Derivatives

Definition

The complex **hyperbolic sine** and **hyperbolic cosine** functions are defined by:

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}.$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z}, \quad \text{and } \operatorname{csch} z = \frac{1}{\sinh z}.$$

Derivatives of Complex Hyperbolic Functions

$$\frac{d}{dz}\sinh z = \cosh z \qquad \frac{d}{dz}\cosh z = \sinh z$$

$$\frac{d}{dz}\tanh z = \operatorname{sech}^2 z \qquad \frac{d}{dz}\coth z = -\operatorname{csch}^2 z$$

$$\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \tanh z \qquad \frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \coth z$$

Proof:

Relations Between Complex Sine/Cosine and Their Hyperbolic Functions

$$\sin z = -i \sinh(iz) \qquad \text{and} \qquad \cos z = \cosh(iz)$$
$$\sinh z = -i \sin(iz) \qquad \text{and} \qquad \cosh z = \cos(iz).$$

$$\tan(iz) = \frac{\sin(iz)}{\cos(iz)} = \frac{i \sinh z}{\cosh z} = i \tanh z.$$

Proof:

EXAMPLE 4 A Hyperbolic Identity

Verify that $\cosh(z_1 + z_1) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$ for all complex z_1 and z_2.

Solution:

Logarithmic Functions

The motivation for the definition of the logarithmic function is based on solving the equation

$$e^w = z$$

for *w*, where *z* is any nonzero complex number.

Definition: Complex Logarithm

Let $z = re^{i\theta}$, the multiple-valued function ln z defined by

$$\ln z = \log_e |z| + i\arg(z) = \log_e r + i(\theta + 2n\pi), \ n = 0, \pm1, \pm2, \ldots$$

Is called the complex logarithm.

Note: The notation ln z will always be used to denote the multiple-valued complex logarithm.

Definition: Principal value of the Logarithm

For $z \neq 0$, we define Log, the principal value of the logarithm, by

$$\text{Ln}\, z = \log_e |z| + i\text{Arg}(z) = \log_e r + i\theta, \ -\pi < \theta \leq \pi$$

Algebraic Properties of Logarithm

Theorem

If z_1 and z_2 are nonzero complex numbers and n is an integer, then

(i) $\ln(z_1 z_2) = \ln z_1 + \ln z_2$

(ii) $\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$

(iii) $\ln z_1^n = n \ln z_1$.

Proof of (i):

$$\begin{aligned}\ln z_1 + \ln z_2 &= \log_e |z_1| + i\arg(z_1) + \log_e |z_2| + i\arg(z_2)\\ &= \log_e |z_1| + \log_e |z_2| + i(\arg(z_1) + \arg(z_2)).\end{aligned}$$

$$\log_e |z_1 z_2| = \log_e |z_1| + \log_e |z_2|.$$

$$\arg(z_1) + \arg(z_2) = \arg(z_1 z_2).$$

$$\ln z_1 + \ln z_2 = \log_e |z_1 z_2| + i\arg(z_1 z_2) = \ln(z_1 z_2).$$

EXAMPLE 5.3 Find the values of $\log(1+i)$ and $\log(i)$.

Solution

EXAMPLE 4 Principal Value of the Complex Logarithm

Compute the principal value of the complex logarithm $\operatorname{Ln} z$ for

(a) $z = i$ (b) $z = 1 + i$ (c) $z = -2$

Solution:

EXAMPLE 2 Solving Trigonometric Equations

Find all solutions to the equation $\sin z = 5$.

Solution:

Analyticity of the Ln Function

- The principal value of the complex logarithm Ln z is discontinuous at the point $z = 0$ since this function is not defined there. This function is also discontinuous at every point on the **negative real axis**.
- The function Ln z is continuous on the set consisting of the complex plane excluding the **non-positive real axis**.
- The real and imaginary parts of Ln z are $u(x,y) = \log_e |z| = \log_e \sqrt{x^2 + y^2}$ and $v(x, y)$ = Arg(z), respectively. From multivariable calculus we have that the function $u(x,y) = \log_e \sqrt{x^2 + y^2}$ is continuous at all points in the plane except (0, 0) and we have that the function $v(x, y)$ = Arg(z) is continuous on the domain $|z| > 0$, $-\pi <$ arg(z) $< \pi$.

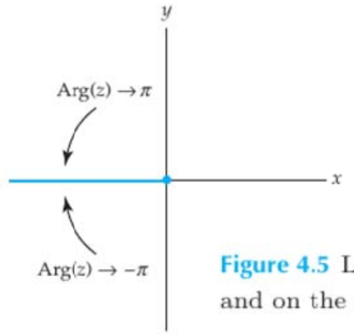


Figure 4.5 Ln z is discontinuous at $z = 0$ and on the negative real axis.

- Thus, the function f_1 defined by
  $f_1 = \log_e r + i\theta$
  is continuous on the domain $|z| > 0$, $-\pi <$ Arg(z) $< \pi$, where $r = |z|$ and θ = Arg(z).
- The function f_1 agrees with the principal value of the complex logarithm Ln z, which is a **branch** of the multiple-valued function $F(z)$ = ln z.

Branch and Branch Cut

Branch: A branch of a multiple-valued function f is any single-valued function F that is analytic in some domain at each point z of which the value $F(z)$ is one of the values of f.

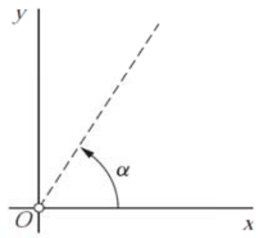
Principal Branch: For each fixed α, the single-valued function

$$\log z = \ln r + i\theta \qquad (r > 0, \alpha < \theta < \alpha + 2\pi),$$

is a branch of the multiple-valued function

$$\log z = \ln r + i(\Theta + 2n\pi) \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

$$\theta = \Theta + 2n\pi \ (n = 0, \pm 1, \pm 2, \ldots), \text{ where } \Theta = \text{Arg } z.$$



The function

$$\text{Log } z = \ln r + i\Theta \qquad (r > 0, -\pi < \Theta < \pi)$$

is called the principal branch.

Branch Cut: A branch cut is a portion of a line or curve that is introduced in order to define a branch F of a multiple-valued function f. Points on the branch cut for F are singular points of F, and any point that is common to all branch cuts of f is called a branch point.

Derivative of $\mathrm{Ln}\, z$

Theorem

The principal branch f_1 of the complex logarithm defined by $f_1(z) = \mathrm{Ln}\, z = \log_e r + i\theta$ is an analytic function and its derivative is given by:

$$\frac{d}{dz}\mathrm{Ln}\, z = f_1'(z) = \frac{1}{z}$$

Proof:

EXAMPLE 5 Derivatives of Logarithmic Functions

Find the derivatives of the following functions in an appropriate domain:

(a) $z\mathrm{Ln}\, z$ and (b) $\mathrm{Ln}(z+1)$.

Solution:

Complex Powers

Definition

General powers of a complex number $z = x + iy$ are defined by the formula

$$z^c = e^{c \ln z} \qquad (c \text{ is complex}, z \neq 0)$$

Since ln z is infinitely many-valued, z^c will, in general, be multivalued. The particular value

$$z^c = e^{c \,\mathrm{Ln}\, z}$$

is called the **principal value** of z^c.

$$\text{P.V. } z^c = e^{c \,\mathrm{Log}\, z}.$$

EXAMPLE 1 Complex Powers

Find the values of the given complex power: **(a)** i^{2i} **(b)** $(1+i)^i$.

Solution:

EX: $(1 + i)^{2-i}$

EXAMPLE 2 **Principal Value of a Complex Power**

Find the principal value of each complex power: **(a)** $(-3)^{i/\pi}$ **(b)** $(2i)^{1-i}$

Solution:

Derivative of Complex Powers

Theorem

(a) On the domain $|z| > 0$, $-\pi < \arg(z) < \pi$, the principal value of the complex power z^α is differentiable and

$$\frac{d}{dz} z^\alpha = \alpha z^{\alpha - 1}.$$

(b) When a value of $\log c$ is specified, where c is any nonzero complex constant, c^z is an entire function of z. In fact,

$$\frac{d}{dz} c^z = \frac{d}{dz} e^{z \cdot \log c} = \log c \cdot e^{z \cdot \log c} = \log c \cdot c^z.$$

EXAMPLE 3 **Derivative of a Power Function**

Find the derivative of the principal value z^i at the point $z = 1 + i$.

Solution:

Inverse Trigonometric Functions

Theorem

$$\sin^{-1} z = -i \log[iz + (1-z^2)^{1/2}].$$
$$\cos^{-1} z = -i \log\left[z + i(1-z^2)^{1/2}\right]$$
$$\tan^{-1} z = \frac{i}{2} \log \frac{i+z}{i-z}.$$

$$\frac{d}{dz}\sin^{-1} z = \frac{1}{(1-z^2)^{1/2}},$$
$$\frac{d}{dz}\cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}.$$
$$\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}$$

Proof of $\sin^{-1} z$**:**

Note: $(1-z^2)^{1/2}$ is, of course, a double-valued function of z.