Unit-2
Complex Integration

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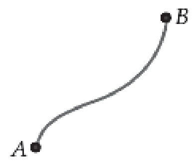
Contour

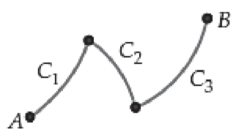
Terminology Suppose a curve C in the plane is parametrized by a set of equations $x = x(t)$, $y = y(t)$, $a \le t \le b$, where $x(t)$ and $y(t)$ are continuous real functions. Let the **initial** and **terminal points** of C, that is, $(x(a), y(a))$ and $(x(b), y(b))$, be denoted by the symbols A and B, respectively. We say that:



(a) Smooth curve and simple

(i) C is a **smooth curve** if x' and y' are continuous on the closed interval $[a, b]$ and not simultaneously zero on the open interval (a, b).

(ii) C is a **piecewise smooth curve** if it consists of a finite number of smooth curves $C_1, C_2, \ldots, C_n$ joined end to end, that is, the terminal point of one curve C_k coinciding with the initial point of the next curve C_{k+1}.



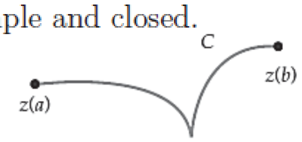
(b) Piecewise smooth curve and simple

(iii) C is a **simple curve** if the curve C does not cross itself except possibly at $t = a$ and $t = b$.

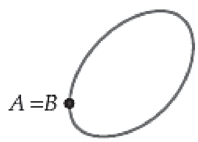
(iv) C is a **closed curve** if $A = B$.

(v) C is a **simple closed curve** if the curve C does not cross itself and $A = B$; that is, C is simple and closed.

(c) Closed but not simple



Curve C is not smooth

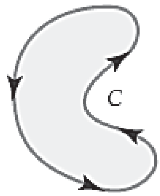


(d) Simple closed curve

Contour

- In complex analysis, a piecewise smooth curve C is called a **contour** or path.

- We define the positive direction on a contour C to be the direction on the curve corresponding to increasing values of the parameter t. It is also said that the curve C has **positive orientation (counterclockwise direction)**.



Positive direction

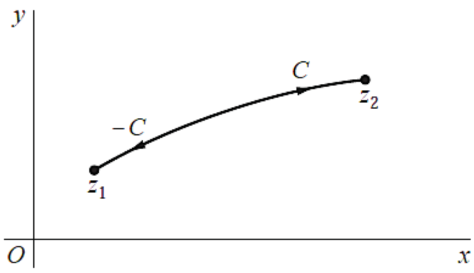
- The **negative direction** on a contour C is the direction opposite the positive direction. If C has an opposite orientation, it is denoted by $-C$. On a simple closed curve, the negative direction corresponds to the **clockwise direction**.

Integral of Complex Function Along a Contour

Definition

Suppose that the equation $z = z(t)$ $(a \le t \le b)$ represents a contour C, extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$. We assume that $f[z(t)]$ is piecewise continuous on the interval $a \le t \le b$ and refer to the function $f(z)$ as being piecewise continuous on C. We then define the line integral, or contour integral, of f along C in terms of the parameter t :

$$\int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt.$$



Note that the integral along $-C$,

$$z = z(-t) \qquad (-b \le t \le -a)$$

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f[z(-t)]\frac{d}{dt}z(-t)\,dt = -\int_{-b}^{-a} f[z(-t)]\,z'(-t)\,dt$$

where $z'(-t)$ denotes the derivative of $z(t)$ with respect to t, evaluated at $-t$. Making the substitution $\tau = -t$ in this last integral, we obtain the expression

$$\int_{-C} f(z)\,dz = -\int_a^b f[z(\tau)]z'(\tau)\,d\tau, \quad \text{this means that} \quad \int_{-C} f(z)\,dz = -\int_C f(z)\,dz.$$

Contour Integral

Suppose that $f(z) = u(z) + iv(z)$ and that $z(t) = x(t) + iy(t)$ is a parametrization for the contour C. Then

$$\begin{aligned}
\int_C f(z)\,dz &= \int_a^b f(z(t))\,z'(t)\,dt \\
&= \int_a^b [u(z(t)) + iv(z(t))]\,[x'(t) + iy'(t)]\,dt \\
&= \int_a^b [u(z(t))\,x'(t) - v(z(t))\,y'(t)]\,dt \\
&\quad + i\int_a^b [v(z(t))\,x'(t) + u(z(t))\,y'(t)]\,dt \\
&= \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy')\,dt,
\end{aligned}$$

$$\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

Properties of Contour Integral

Suppose the functions f and g are continuous in a domain D, and C is a smooth curve lying entirely in D. Then

(i) $\int_C kf(z)\,dz = k\int_C f(z)\,dz$, k a complex constant.

(ii) $\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$.

(iii) $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$, where C consists of the smooth curves C_1 and C_2 joined end to end.

(iv) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$, where $-C$ denotes the curve having the opposite orientation of C.

Line Integral of a General Complex Function

Dependence on path.

If we integrate a given function $f(z)$ from a point z_0 to a point z_1 along different paths, the integrals will in general have different values. In other words, a complex line integral depends not only on the endpoints of the path but in general also on the path itself.

Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \leqq t \leqq b$. Let $f(z)$ be a continuous function on C. Then

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\dot{z}(t)\,dt \qquad \left(\dot{z} = \frac{dz}{dt}\right).$$

Steps in Calculation:

(A) Represent the path C in the form $z(t)$ $(a \leqq t \leqq b)$.

(B) Calculate the derivative $\dot{z}(t) = dz/dt$.

(C) Substitute $z(t)$ for every z in $f(z)$ (hence $x(t)$ for x and $y(t)$ for y).

(D) Integrate $f[z(t)]\dot{z}(t)$ over t from a to b.

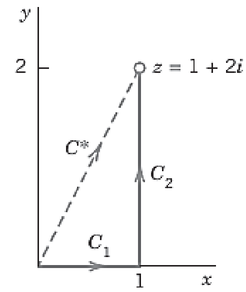
Ex 1:

Evaluate **(a)** $\int_C xy^2 dx$, **(b)** $\int_C xy^2 dy$, and **(c)** $\int_C xy^2 ds$, where the path of integration C is the quarter circle defined by $x = 4\cos t$, $y = 4\sin t$, $0 \leq t \leq \pi/2$.

Solution

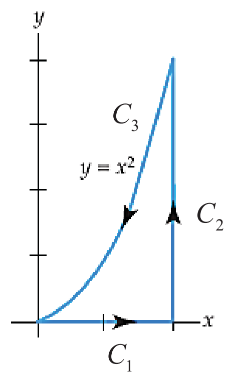
Ex 2: Integrate $f(z) = \text{Re } z = x$ from 0 to $1 + 2i$ (a) along C^*, (b) along C consisting of C_1 and C_2.

Solution.



Ex 3: Evaluate $\oint_C y^2\,dx - x^2 dy$, where C is the closed curve

Solution:



$C = C_1 + C_2 + C_3$

ML Inequality

If f is continuous on a smooth curve C and if $|f(z)| \le M$ for all z on C, then $\left|\int_C f(z)\,dz\right| \le ML$, where L is the length of C.

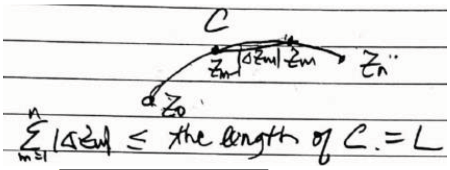
Proof:

The complex integral of f on C is

$$\int_C f(z)\,dz = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(z_k^*)\,\Delta z_k.$$

It follows from the form of the triangle inequality

$$\left|\sum_{k=1}^{n} f(z_k^*)\Delta z_k\right| \le \sum_{k=1}^{n} |f(z_k^*)|\,|\Delta z_k| \le M \sum_{k=1}^{n} |\Delta z_k|.$$



Because $|\Delta z_k| = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$, we can interpret $|\Delta z_k|$ as the length of the chord joining the points z_k and z_{k-1} on C. Moreover, since the sum of the lengths of the chords cannot be greater than the length L of C, the inequality (14) continues as $|\sum_{k=1}^{n} f(z_k^*)\Delta z_k| \le ML$. Finally, the continuity of f guarantees that $\int_C f(z)\,dz$ exists, and so if we let $\|P\| \to 0$, the last inequality yields $\left|\int_C f(z)\,dz\right| \le ML$.

Ex 1:

$$\left|\int_c \frac{1}{z^2+1}\,dz\right| \le \frac{1}{2\sqrt{5}},$$

where C is the straight-line segment from 2 to $2+i$.

Sol:

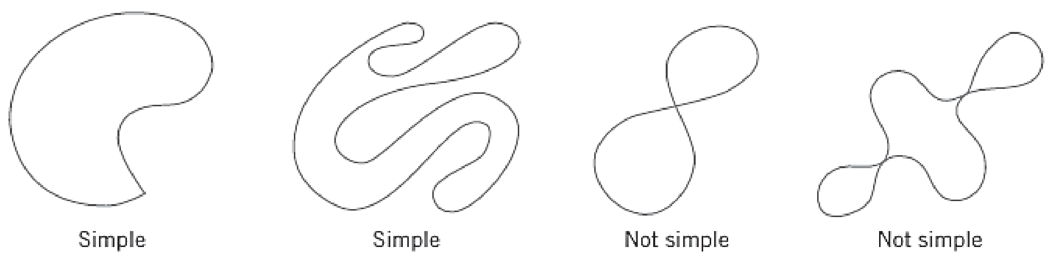
Ex 2:

Find an upper bound for the absolute value of $\oint_C \frac{e^z}{z+1}\,dz$ where C is the circle $|z| = 4$.

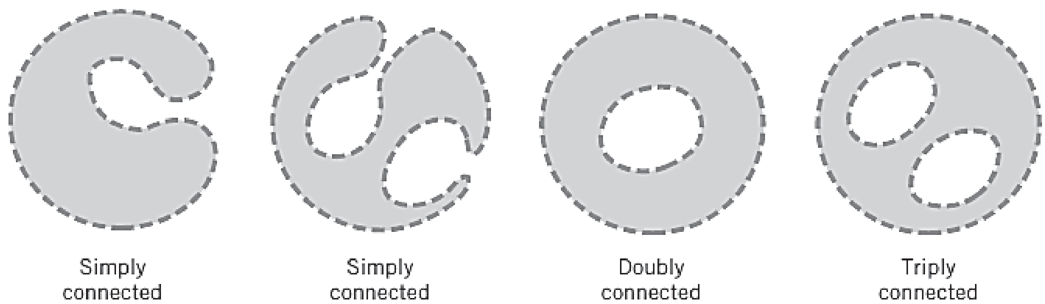
Sol:

Topology of Paths

- **Simple Closed Path**



- **Simply Connected Domain**: A simply connected domain is a path-connected domain where one can continuously shrink any simple closed curve into a point while remaining in the domain.



Green's Theorem

Theorem (**Green's theorem**):

Let C be a simple closed contour with positive orientation and let R be the domain that forms the interior of C. If P and Q are continuous and have continuous partial derivatives P_x , P_y , Q_x , and Q_y at all points on C and R, then

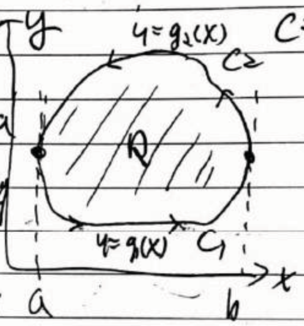
$$\int_C P(x,y)dx + Q(x,y)dy = \iint_R [Q_x(x,y) - P_y(x,y)]dxdy$$

Proof:

(i) Consider $a \le x \le b$

$C_1: z_1(t) = t + i g_1(t), \ a \le t \le b$

$C_2: z_2(t) = -t + i g_2(-t), \ -b \le t \le -a$



$$\iint_R P_y(x,y)\,dxdy = \int_a^b \int_{g_1(x)}^{g_2(x)} P_y(x,y)\,dydx$$

$$= \int_a^b P(x,g_2(x))dx - \int_a^b P(x,g_1(x))dx$$

$$= \int_{-b}^{-a} P(-t, g_2(-t))(-1)dt - \int_a^b P(t, g_1(t))dt$$

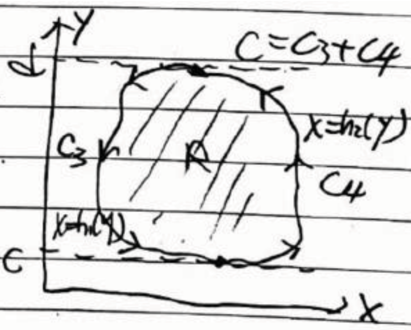
$$= -\int_{C_2} P(x,y)dx - \int_{C_1} P(x,y)dx = -\int_C P(x,y)dx$$

Proof of Green's Theorem

(ii) Consider $c \le y \le d$

$C_3: z_3(t) = h_1(-t) - it, \ -d \le t \le -c$

$C_4: z_4(t) = h_2(t) + it, \ c \le t \le d$



$$\iint_R Q_x(x,y)\,dxdy$$

$$= \int_c^d \int_{h_1(y)}^{h_2(y)} Q_x(x,y)\,dxdy.$$

$$= \int_c^d Q(h_2(y),y)dy - \int_c^d Q(h_1(y),y)dy$$

$$= \int_c^d Q(h_2(t),t)dt - \int_c^d Q(h_1(-t),-t)(-1)dt.$$

$$= \int_{C_4} Q(x,y)dy + \int_{C_3} Q(x,y)dy = \int_C Q(x,y)dy$$

$$\Rightarrow \oint_C P(x,y)dx + Q(x,y)dy = \iint_R [Q_x(x,y) - P_y(x,y)]dxdy$$

#

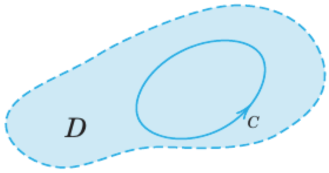
Cauchy Integral Theorem

Suppose that a function f is analytic in a simply connected domain D and that f' is continuous in D. Then for every simple closed contour C in D, $\oint_C f(z)\,dz = 0$.

Proof:

$$\oint_C f(z)\,dz = \oint_C u(x,y)\,dx - v(x,y)\,dy + i\oint_C v(x,y)\,dx + u(x,y)\,dy$$

With Green's theorem $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$



$$\oint_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA.$$

Because f is analytic in D, the real functions u and v satisfy the Cauchy-Riemann equations, $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$, at every point in D. Using the Cauchy-Riemann equations to replace $\partial u/\partial y$ and $\partial u/\partial x$ shows that

$$\oint_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dA + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dA$$
$$= \iint_R (0)\; dA + i\iint_R (0)\; dA = 0.$$

Example 1:

Entire Functions

$$\oint_C e^z\,dz = 0, \qquad \oint_C \cos z\,dz = 0, \qquad \oint_C z^n\,dz = 0 \qquad (n = 0, 1, \cdots)$$

for any closed path, since these functions are entire (analytic for all z).

EXAMPLE 2 **Applying the Cauchy-Goursat Theorem**

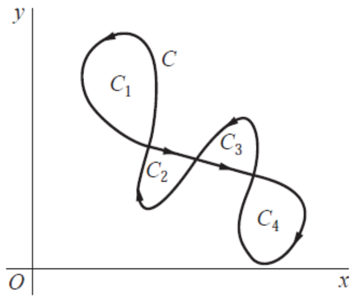
Evaluate $\oint_C \frac{dz}{z^2}$, where the contour C is the ellipse $(x-2)^2 + \frac{1}{4}(y-5)^2 = 1$.

Solution

Closed Contour with Self-intersection Points

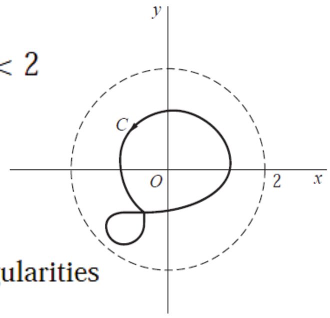
- If f is analytic at each point interior to and on C,

$$\int_C f(z)\,dz = \sum_{k=1}^{4} \int_{C_k} f(z)\,dz = 0.$$



Example: If C denotes any closed contour lying in the open disk $|z| < 2$

$$\int_C \frac{z\,e^z}{(z^2+9)^5}\,dz = 0.$$



This is because the disk is a simply connected domain and the two singularities $z = \pm 3i$ of the integrand are exterior to the disk.

Independence of Path

Independence of Path

If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.

Proof:

Line Integral of an Analytic Complex Function

Independence of Path.

Let $f(z)$ be analytic in a simply connected domain D. Then there exists an indefinite integral of $f(z)$ in the domain D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_1 in D we have

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)].$$

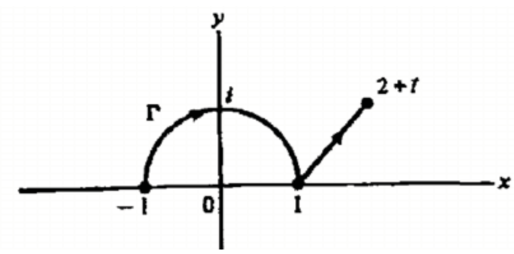
(Note that we can write z_0 and z_1 instead of C, since we get the same value for all those C from z_0 to z_1.)

Proof:

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt = \int_a^b F'(z(t))z'(t)\,dt$$
$$= \int_a^b \frac{d}{dt}F(z(t))\,dt \quad \leftarrow \text{chain rule}$$
$$= F(z(t))\,|_a^b$$
$$= F(z(b)) - F(z(a)) = F(z_1) - F(z_0).$$

Line Integration of Analytic Functions

Example: Compute the integral $\int_\Gamma \cos z \; dz$.



Solution:

Contour Integration of Non-analytic Functions

Example

$$\oint_C \bar{z}\,dz = \int_0^{2\pi} e^{-it} i e^{it}\,dt = 2\pi i$$

where C: $z(t) = e^{it}$ is the unit circle. This does not contradict Cauchy's theorem because $f(z) = \bar{z}$ is not analytic.

Contour Integration of Not Simply Connected (Doubly Connected) Functions

Corollary: Let z_0 denote a fixed complex value. If C is a simple closed contour with positive orientation such that z_0 lies interior to C, then

$$\oint_C \frac{dz}{z - z_0} = 2\pi i \qquad \text{and} \qquad \oint_C \frac{dz}{(z - z_0)^m} = 0,$$

where m is any number except $m = 1$.

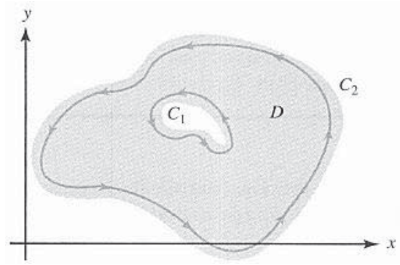
Solution.

Deformation of Contour

Theorem (**Deformation of Contour**):

Let C_1 and C_2 be two simple closed positively oriented contours such that C_1 lies interior to C_2. If f is analytic in a domain D that both C_1 and C_2 are the region between them, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

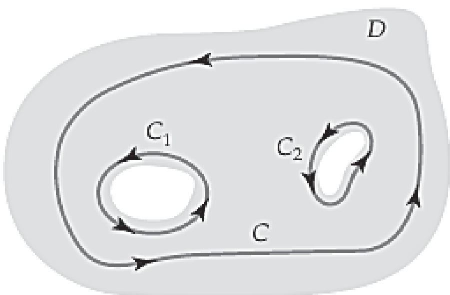


Proof:

Multiply Connected Domains

Suppose $C, C_1, \ldots, C_n$ are simple closed curves with a positive orientation such that $C_1, C_2, \ldots, C_n$ are interior to C but the regions interior to each C_k, $k = 1, 2, \ldots, n$, have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the C_k, $k = 1,\ 2,\ \ldots,\ n$, then

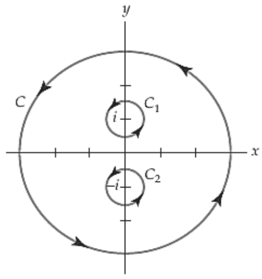
$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz.$$



Ex 1: Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}dz$, where C is circle $|z-2|=2$.

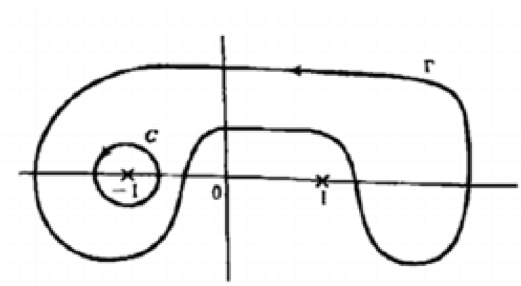
Sol:

EX 2: Evaluate $\oint_C \frac{dz}{z^2+1}$, where C is the circle $|z|=4$.



Sol:

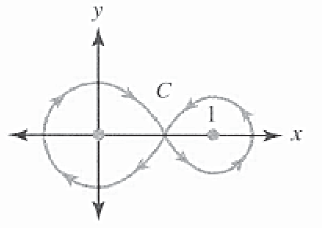
EX 3: Evaluate $\int_\Gamma 1/(z^2-1)dz$, where Γ is depicted as below.



Sol:

EX 4: Show that $\int_C \frac{z-2}{z^2-z}\, dz = -6\pi i$, where C is the "figure eight" contour

Sol:



(a) The figure eight contour C.

Cauchy Integral Formula

Theorem. *Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{z - z_0}.$$

Proof:

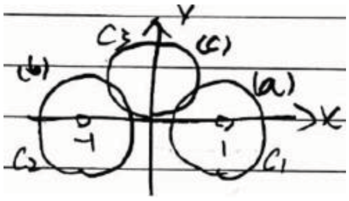
EX 1: Evaluate $\oint_C \frac{z^2 - 4z + 4}{z + i}\, dz$, where C is the circle $|z| = 2$.

Solution

EX 2: Evaluate $\oint_C \frac{z}{z^2 + 9}\, dz$, where C is the circle $|z - 2i| = 4$.

Solution

EX 3: Integrate $g(z) = \frac{z^2 + 1}{z^2 - 1}$ counterclockwise around (a), (b) and (c) contours.

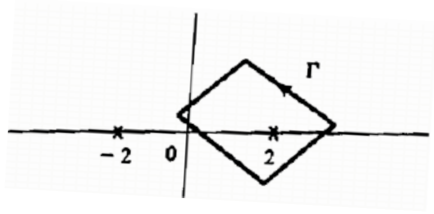


Sol:

Ex 4: Evaluate the integral

$$\oint_{\Gamma} \frac{\cos z}{z^2 - 4} dz$$

along the contour Γ.



EX 5: Compute

$$\oint_{C} \frac{z^2 e^z}{2z + i} dz$$

where C is the unit circle $|z|=1$ traversed in the clockwise direction.

Sol:

Exercise: Compute (in the counterclockwise direction)

$$\oint_C \frac{z^2 + 3z + 2i}{z^2 + 3z - 4}\, dz;$$

for (a) C: |z|=2, (b) C: |z+5|=2, and (c) |z|=5.

Ans: (a) $(8\pi i - 4\pi)/5$; (b) $-(8\pi i - 4\pi)/5$; (c) 0

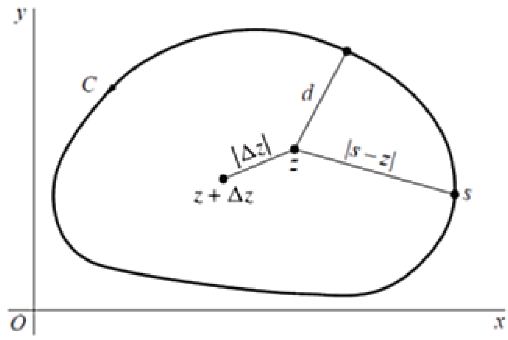
Extension of the Cauchy Integral Formula

Verify that

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}$$

where z is interior to C and where s denotes points on C.

Sol:



Cauchy's Integral Formula for Derivatives

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D. The values of these derivatives at a point z_0 in D are given by the formulas

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,dz$$

and in general

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad (n = 1, 2, \cdots);$$

here C is any simple closed path in D that encloses z_0 and whose full interior belongs to D; and we integrate counterclockwise around C

EX 1: Contour C encloses πi in counterclockwise sense,

$$\oint_C \frac{\cos z}{(z-\pi i)^2}\,dz =$$

EX2: Contour C encloses $-i$ in counterclockwise sense,

$$\oint_C \frac{z^4-3z^2+6}{(z+i)^3}\,dz =$$

EX3: Contour C encloses 1 and $\pm 2i$ lies outside in counterclockwise sense,

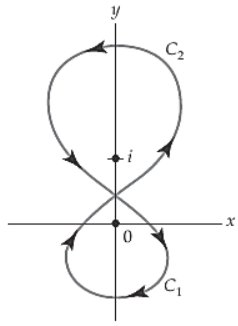
$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)}\,dz =$$

EX 4: Evaluate $\oint_C \frac{z+1}{z^4+2iz^3}\,dz$, where C is the circle $|z|=1$.

Solution

EX 5: Evaluate $\displaystyle\int_C \frac{z^3+3}{z(z-i)^2}\,dz$, where C is the figure-eight contour

Sol:



Cauchy's Inequality

Suppose that f is analytic in a simply connected domain D and C is a circle defined by $|z-z_0|=r$ that lies entirely in D. If $|f(z)| \le M$ for all points z on C, then

$$\left|f^{(n)}(z_0)\right| \le \frac{n!M}{r^n}.$$

Proof: