Unit-3
Complex Series

張大中
中央大學 通訊工程系
dcchang@ce.ncu.edu.tw

Convergence Sequence

Convergence:

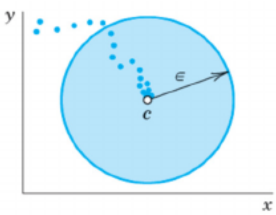
A **convergent sequence** $z_1, z_2, \cdots$ is one that has a limit c, written

$$\lim_{n\to\infty} z_n = c \qquad \text{or simply} \qquad z_n \to c.$$

Mathematical Definition: For every $\epsilon > 0$, we can find an N such that

$$|z_n - c| < \epsilon \qquad \text{for all} \quad n>N$$

Geometrically, all terms with lie in the open disk of radius ϵ and center c and only finitely many terms do not lie in that disk.



Convergent complex sequence

Cauchy's Convergence Principle for Series

A series $z_1 + z_2 + \cdots$ is convergent if and only if for every given $\epsilon > 0$ (no matter how small) we can find an N (which depends on ϵ, in general) such that

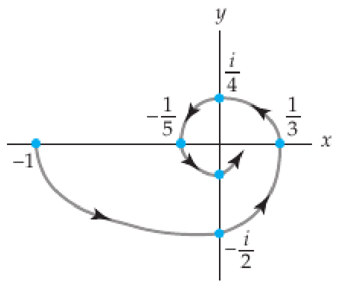
$$(5) \quad |z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \epsilon \qquad \text{for every } n > N \text{ and } p = 1, 2, \cdots$$

Example:

The sequence $\left\{ \frac{i^{n+1}}{n} \right\}$ converges since $\lim_{n\to\infty} \frac{i^{n+1}}{n} = 0$. As we see from

$$-1, -\frac{i}{2}, \frac{1}{3}, \frac{i}{4}, -\frac{1}{5}, \cdots,$$

the terms of the sequence, marked by colored dots in the figure, spiral in toward the point $z = 0$ as n increases.



Criterion for Convergence

A sequence $\{z_n\}$ converges to a complex number $L = a + ib$ if and only if $\mathrm{Re}(z_n)$ converges to $\mathrm{Re}(L) = a$ and $\mathrm{Im}(z_n)$ converges to $\mathrm{Im}(L) = b$.

Example:

Consider the sequence $\left\{ \frac{3+ni}{n+2ni} \right\}$. From

$$z_n = \frac{3+ni}{n+2ni} = \frac{(3+ni)(n-2ni)}{n^2+4n^2} = \frac{2n^2+3n}{5n^2} + i\frac{n^2-6n}{5n^2},$$

we see that

$$\mathrm{Re}(z_n) = \frac{2n^2+3n}{5n^2} = \frac{2}{5} + \frac{3}{5n} \to \frac{2}{5}$$

and $$\mathrm{Im}(z_n) = \frac{n^2-6n}{5n^2} = \frac{1}{5} - \frac{6}{5n} \to \frac{1}{5}$$

the last results are sufficient for us to conclude that the given sequence converges to $a + ib = \frac{2}{5} + \frac{1}{5}i$.

Convergent Series

- **Convergent series**

Given a sequence $z_1, z_2, \cdots, z_n, \cdots$, in general,

$$s_n = z_1 + z_2 + \cdots + z_n$$

s_n is called the nth partial sum of the infinite series. A **convergent series** is one whose sequence of partial sums converges, say,

$$\lim_{n\to\infty} s_n = s. \qquad \text{Then we write} \qquad s = \sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$$

and call *s* the sum or value of the series. A series that is not convergent is called a **divergent series**.

- **Divergence Theorem**

If a series $z_1 + z_2 + \cdots$ converges, then $\lim_{m\to\infty} z_m = 0$. Hence if this does not hold, the series diverges.

Proof

Absolute and Conditional Convergence

An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be **absolutely convergent** if $\sum_{k=1}^{\infty} |z_k|$ converges. An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be **conditionally convergent** if it converges but $\sum_{k=1}^{\infty} |z_k|$ diverges.

Remark: The absolute convergence of a series of complex numbers implies the convergence of that series.

Note: series converges $\Rightarrow \lim_{m\to\infty} z_m = 0$

$\lim_{m\to\infty} z_m = 0 \not\Rightarrow$ series converges.

Ex: Series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ does not converges.

Series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges.

Geometric Series

Geometric Series

*The **geometric series***

$$\sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \cdots$$

converges with the sum $1/(1-q)$ *if* $|q| < 1$ *and diverges if* $|q| \geqq 1$.

Proof:

Example:

The infinite series

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{1+2i}{5} + \frac{(1+2i)^2}{5^2} + \frac{(1+2i)^3}{5^3} + \cdots$$

is a geometric series. It has the form $z = \frac{1}{5}(1+2i)$.
Since $|z| = \sqrt{5}/5 < 1$, the series is convergent and its sum is

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{\frac{1+2i}{5}}{1 - \frac{1+2i}{5}} = \frac{1+2i}{4-2i} = \frac{1}{2}i.$$

Comparison Test

If a series $z_1 + z_2 + \cdots$ *is given and we can find a convergent series* $b_1 + b_2 + \cdots$ *with nonnegative real terms such that* $|z_1| \leqq b_1, |z_2| \leqq b_2, \cdots$, *then the given series converges, even absolutely.*

Proof:

By Cauchy's principle, since $b_1 + b_2 + \cdots$ converges, for any given $\epsilon > 0$ we can find an N such that

$$b_{n+1} + \cdots + b_{n+p} < \epsilon \qquad \text{for every } n > N \text{ and } p = 1, 2, \cdots.$$

From this and $|z_1| \leqq b_1, |z_2| \leqq b_2, \cdots$ we conclude that for those n and p,

$$|z_{n+1}| + \cdots + |z_{n+p}| \leqq b_{n+1} + \cdots + b_{n+p} < \epsilon.$$

Hence, again by Cauchy's principle, $|z_1| + |z_2| + \cdots$ converges, so that $z_1 + z_2 + \cdots$ is absolutely convergent. ■

Example: Show that the following series converges.

$$\sum_{n=0}^{\infty} \frac{3+2i}{(n+1)^n}$$

Sol:

Ratio Test

If a series $z_1 + z_2 + \cdots$ with $z_n \neq 0$ $(n = 1, 2, \cdots)$ has the property that for every n greater than some N,

$$\left|\frac{z_{n+1}}{z_n}\right| \leqq q < 1 \qquad (n > N)$$

(where $q < 1$ is fixed), this series converges absolutely. If for every $n > N$,

$$\left|\frac{z_{n+1}}{z_n}\right| \geqq 1 \qquad (n > N),$$

the series diverges.

Limitation Version of Ratio Test

If a series $z_1 + z_2 + \cdots$ with $z_n \neq 0$ $(n = 1, 2, \cdots)$ is such that $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L$, then:

(a) *If $L < 1$, the series converges absolutely.*

(b) *If $L > 1$, the series diverges.*

(c) *If $L = 1$, the series may converge or diverge, so that the test fails and permits no conclusion.*

Remark:

(c) The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ has $z_{n+1}/z_n = n/(n+1)$, hence $L = 1$, and diverges. The series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots \qquad \text{has} \qquad \frac{z_{n+1}}{z_n} = \frac{n^2}{(n+1)^2},$$

hence also $L = 1$, but it converges. Convergence follows from (Fig. 364)

$$s_n = 1 + \frac{1}{4} + \cdots + \frac{1}{n^2} \leqq 1 + \int_1^n \frac{dx}{x^2} = 2 - \frac{1}{n},$$

so that $s_1, s_2, \cdots$ is a bounded sequence and is monotone increasing (since the terms of the series are all positive); both properties together are sufficient for the convergence of the real sequence $s_1, s_2, \cdots$.

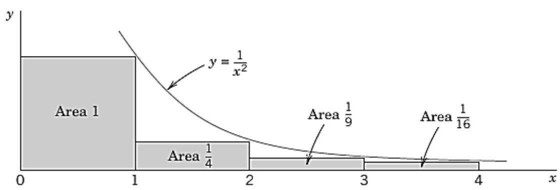


Fig. 364. Convergence of the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

Proof

Note: harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots>1+\left(\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots$$
$$=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots=\infty$$

Note: $\sum_{n=1}^{N}\frac{1}{n^2}<1+\sum_{n=2}^{N}\frac{1}{n(n-1)}=1+\sum_{n=2}^{N}\left(\frac{1}{n-1}-\frac{1}{n}\right)=1+1-\frac{1}{N}<2 \text{ as } N\to\infty.$

Example: Show that $\sum_{n=1}^{\infty}\frac{(3+4i)^n}{(5^n n^2)}$ converges.

EX 1: Is the following series convergent or divergent?

$$\sum_{n=0}^{\infty}\frac{(100+75i)^n}{n!}=1+(100+75i)+\frac{1}{2!}(100+75i)^2+\cdots$$

Sol:

EX 2: Show that $\sum_{n=0}^{\infty}\frac{(1-i)^n}{n!}$ converges.

Sol:

EX3: Let $a_n = i/2^{3n}$ and $b_n = 1/2^{3n+1}$. Is the following series convergent or divergent?

$$a_0 + b_0 + a_1 + b_1 + \cdots = i + \frac{1}{2} + \frac{i}{8} + \frac{1}{16} + \frac{i}{64} + \frac{1}{128} + \cdots$$

Sol:

EX 4: Show that the series $\sum_{n=0}^{\infty} \frac{(z-i)^n}{2^n}$ converges for all values of z in the disk $|z-i| < 2$ and diverges if $|z-i| > 2$.

Sol:

Root Test

If a series $z_1 + z_2 + \cdots$ is such that for every n greater than some N,

$$\sqrt[n]{|z_n|} \leqq q < 1 \qquad (n > N) \tag{9}$$

(where $q < 1$ is fixed), this series converges absolutely. If for infinitely many n,

$$\sqrt[n]{|z_n|} \geqq 1, \tag{10}$$

the series diverges.

Proof:

If (9) holds, then $|z_n| \leqq q^n < 1$ for all $n > N$. Hence the series $|z_1| + |z_2| + \cdots$ converges by comparison with the geometric series, so that the series $z_1 + z_2 + \cdots$ converges absolutely. If (10) holds, then $|z_n| \geqq 1$ for infinitely many n. Divergence of $z_1 + z_2 + \cdots$ now follows from the **Divergence Theorem**. ■

Remark: limitation version of geometric series

Root Test

If a series $z_1 + z_2 + \cdots$ is such that $\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$, then:

(a) *The series converges absolutely if $L < 1$.*

(b) *The series diverges if $L > 1$.*

(c) *If $L = 1$, the test fails; that is, no conclusion is possible.*

Power Series

A **power series** *in powers of* $z - z_0$ is a series of the form

$$\text{(1)} \qquad \sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

where z is a complex variable, $a_0, a_1, \cdots$ are complex (or real) constants, called the **coefficients** of the series, and z_0 is a complex (or real) constant, called the **center** of the series. This generalizes real power series of calculus.

If $z_0 = 0$, we obtain as a particular case a *power series in powers of* z:

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots.$$

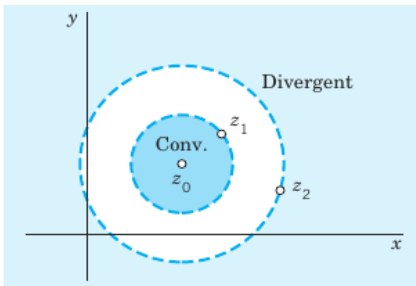
Convergence of a Power Series

(a) *Every power series* (1) *converges at the center* z_0.

(b) *If* (1) *converges at a point* $z = z_1 \neq z_0$, *it converges absolutely for every* z *closer to* z_0 *than* z_1, *that is,* $|z - z_0| < |z_1 - z_0|$.

(c) *If* (1) *diverges at* $z = z_2$, *it diverges for every* z *farther away from* z_0 *than* z_2

Proof:

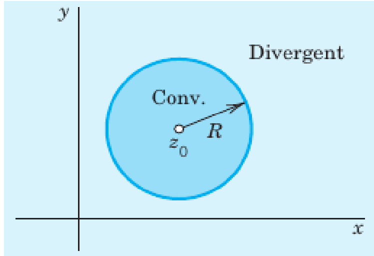


Circle of Convergence

If a power series converges everywhere within a circle for all *z* for which

$$|z - z_0| < R$$

then $|z - z_0|$ is called the circle of convergence and *R* the radius of convergence.



Example 1:

Behavior on the Circle of Convergence

On the circle of convergence (radius $R = 1$ in all three series),

$\Sigma\, z^n/n^2$ converges everywhere since $\Sigma\, 1/n^2$ converges,

$\Sigma\, z^n/n$ converges at -1 (by Leibniz's test) but diverges at 1,

$\Sigma\, z^n$ diverges everywhere.

Notations $R = \infty$ and $R = 0$. To incorporate these two excluded cases in the present notation, we write

$R = \infty$ if the series (1) converges for all z (as in Example 2),
$R = 0$ if (1) converges only at the center $z = z_0$ (as in Example 3).

Example 2

The power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

is absolutely convergent for every z. In fact, by the ratio test, for any fixed z,

$$\left| \frac{z^{n+1}/(n+1)!}{z^n/n!} \right| = \frac{|z|}{n+1} \quad \to \quad 0 \quad \text{as} \quad n \to \infty.$$

Example 3

The following power series converges only at $z = 0$, but diverges for every $z \neq 0$, as we shall show.

$$\sum_{n=0}^{\infty} n! z^n = 1 + z + 2z^2 + 6z^3 + \cdots$$

In fact, from the ratio test we have

$$\left| \frac{(n+1)! z^{n+1}}{n! z^n} \right| = (n+1)|z| \quad \to \quad \infty \quad \text{as} \quad n \to \infty \quad (z \text{ fixed and } \neq 0).$$

Radius of Convergence R

Theorem

Suppose that the sequence $|a_{n+1}/a_n|, n = 1, 2, \cdots,$ *converges with limit* L^*. *If* $L^* = 0$, *then* $R = \infty$; *that is, the power series* (1) *converges for all* z. *If* $L^* \neq 0$ *(hence* $L^* > 0$*), then*

$$R = \frac{1}{L^*} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

If $|a_{n+1}/a_n| \to \infty$, *then* $R = 0$ *(convergence only at the center* z_0*).*

Proof:

Ex 1: Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(z - 3i)^n$

Sol:

EX 2: Find the radius of convergence of the power series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}(z - 1 - i)^k$.

Sol:

Radius of Convergence for Root Test

For a power series $\sum_{k=0}^{\infty} a_k (z-z_0)^k$, the radius of convergence for root test is

$$R = 1/L, \text{ where } L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

Proof:

Example:

Consider the power series $\sum_{k=1}^{\infty} \left(\frac{6k+1}{2k+5}\right)^k (z-2i)^k$.

Ex: Find the circle and radius of convergence of the following power series:

$$\sum_{k=1}^{\infty} (-1)^k \left(\frac{1+2i}{2}\right)^k (z+2i)^k$$

Sol:

Uniqueness of Power Series

Theorem

Let the power series $a_0 + a_1 z + a_2 z^2 + \cdots$ and $b_0 + b_1 z + b_2 z^2 + \cdots$ both be convergent for $|z| < R$, where R is positive, and let them both have the same sum for all these z. Then the series are identical, that is, $a_0 = b_0, a_1 = b_1, a_2 = b_2, \cdots$.

Hence if a function $f(z)$ can be represented by a power series with any center z_0, this representation is ***unique****.*

Proof:

Term-by-Term Differentiation of Power Series

Theorem

A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ can be differentiated term by term within its circle of convergence $|z - z_0| = R$.

Proof:

Differentiating a power series term-by-term gives,

$$\frac{d}{dz}\sum_{k=0}^{\infty} a_k (z - z_0)^k = \sum_{k=0}^{\infty} a_k \frac{d}{dz}(z - z_0)^k = \sum_{k=1}^{\infty} a_k k (z - z_0)^{k-1}.$$

Note that the summation index in the last series starts with $k = 1$ because the term corresponding to $k = 0$ is zero. It is readily proved by the ratio test that the original series and the differentiated series,

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad \text{and} \qquad \sum_{k=1}^{\infty} a_k k (z - z_0)^{k-1}$$

have the same circle of convergence $|z - z_0| = R$. Since the derivative of a power series is another power series, the first series $\sum_{k=1}^{\infty} a_k (z - z_0)^k$ can be differentiated as many times as we wish. In other words, it follows that *a power series defines an infinitely differentiable function* within its circle of convergence and each differentiated series has the same radius of convergence R as the original power series.

Integration of Power Series

Theorem

A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ can be integrated term-by-term within its circle of convergence $|z-z_0| = R$, for every contour C lying entirely within the circle of convergence.

Proof:

The theorem states that

$$\int_C \sum_{k=0}^{\infty} a_k(z-z_0)^k\,dz = \sum_{k=0}^{\infty} a_k \int_C (z-z_0)^k\,dz$$

whenever C lies in the interior of $|z-z_0| = R$. Indefinite integration can also be carried out term by term:

$$\int \sum_{k=0}^{\infty} a_k(z-z_0)^k dz = \sum_{k=0}^{\infty} a_k \int (z-z_0)^k dz$$

$$= \sum_{k=0}^{\infty} \frac{a_k}{k+1}(z-z_0)^{k+1} + \text{constant}.$$

The ratio test can be used to be prove that both

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{a_k}{k+1}(z-z_0)^{k+1}$$

have the same circle of convergence $|z-z_0| = R$.

Multiplication of Power Series (Cauchy Product)

Suppose that each of the power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n(z-z_0)^n$$

converges within some circle $|z-z_0| = R$. Their sums $f(z)$ and $g(z)$, respectively, are then analytic functions in the disk $|z-z_0| < R$, and the product of those sums is valid there:

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n \qquad (|z-z_0| < R).$$

Proof:

$$c_0 = f(z_0)g(z_0) = a_0b_0,$$

$$c_1 = \frac{f(z_0)g'(z_0) + f'(z_0)g(z_0)}{1!} = a_0b_1 + a_1b_0,$$

$$c_2 = \frac{f(z_0)g''(z_0) + 2f'(z_0)g'(z_0) + f''(z_0)g(z_0)}{2!} = a_0b_2 + a_1b_1 + a_2b_0.$$

The general expression for any coefficient c_n is easily obtained

$$[f(z)g(z)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z)g^{(n-k)}(z) \qquad (n = 1, 2, \ldots), \quad \text{where } \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\Rightarrow \quad c_n = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} \cdot \frac{g^{(n-k)}(z_0)}{(n-k)!} = \sum_{k=0}^{n} a_k b_{n-k};$$

Example: Find the power series of $e^z/(1+z)$ in the open disk $|z|<1$.

Sol:

Division of Power Series

Suppose that each of the power series

(1) $$\sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n(z-z_0)^n$$

converges within some circle $|z-z_0|=R$.

Continuing to let $f(z)$ and $g(z)$ denote the sums of series (1), suppose that $g(z)\neq 0$ when $|z-z_0|<R$. Since the quotient $f(z)/g(z)$ is analytic throughout the disk $|z-z_0|<R$, it has a power series representation

$$\frac{f(z)}{g(z)}=\sum_{n=0}^{\infty} d_n(z-z_0)^n \qquad (|z-z_0|<R),$$

where the coefficients d_n can be found by differentiating $f(z)/g(z)$ successively and evaluating the derivatives at $z=z_0$. The results are the same as those found by formally carrying out the division of the first of series (1) by the second.

Example: Find the power series of

$$\frac{1}{z^2 \sinh z} \qquad 0 < |z| < \pi .$$

Sol:

Taylor's Theorem

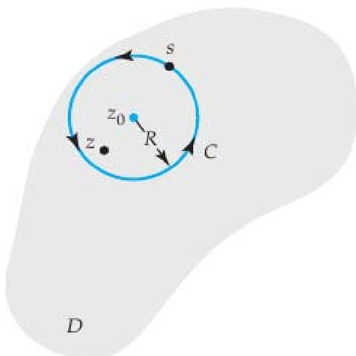
Theorem

Let f be analytic within a domain D and let z_0 be a point in D. Then f has the series representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

valid for the largest circle C with center at z_0 and radius R that lies entirely within D.

Proof:



Proof of $|R_n(z)| \to 0$ as $n \to \infty$

Proof of $|R_n(z)| \to 0$ as $n \to \infty$:

Maclaurin Series

Definition: A Taylor series with center $z_0 = 0$,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k,$$

is referred to as a **Maclaurin series**.

Some Important Maclaurin Series

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

Example: Find the Maclaurin series of $\tan z$.

Sol:

Radius of Convergence for a Taylor Series

- We can find the radius of convergence of a Taylor series in exactly with the ratio test or the root test.
- However, we can simplify matters even further by noting that the *radius of convergence R is the distance from the center z_0 of the series to the nearest* ***isolated singularity*** *of f*.

Remark: An ***isolated singularity*** is a point at which f fails to be analytic but is, nonetheless, analytic at all other points throughout some neighborhood of the point.

Example:

Suppose the function $f(z) = \dfrac{3-i}{1-i+z}$ is expanded in a Taylor series with center $z_0 = 4-2i$. What is its radius of convergence R?

Solution

Theoretical Method for Taylor Series

EX 1: Find the Maclaurin expansion of $f(z) = 1/(1 - z)$.

Sol:

EX 2: Find the Maclaurin expansion of e^z .

Sol:

EX 3: Find the Maclaurin expansion of $\sin z$.

Sol:

EX 4: Find the Maclaurin expansion of $\sinh z$.

Sol:

EX 5: Find the Maclaurin expansion of $\mathrm{Ln}(1+z)$.

Sol:

Substitution Method (Uniqueness of Power Series) for Taylor Series

EX 1: Find the Maclaurin expansion of $f(z) = 1/(1+z^2)$.

Sol:

EX 2: Find the Maclaurin expansion of $f(z) = \arctan z$.

Sol:

EX 3: Find the Maclaurin expansion of $f(z) = \dfrac{1+2z^2}{z^3+z^5}$

Sol:

EX 4: Find the Maclaurin expansion of $f(z) = \dfrac{1}{(1-z)^2}$.

Sol:

Note: Maclaurin expansion of $f(z) = \dfrac{z^3}{(1-z)^2}$,

$$\frac{z^3}{(1-z)^2} = z^3 + 2z^4 + 3z^5 + \cdots = \sum_{k=1}^{\infty} k\, z^{k+2}.\quad \text{the radius of convergence } R = 1.$$

EX 5: Expand $f(z) = \dfrac{1}{1-z}$ in a Taylor series with center $z_0 = 2i$.

Sol:

Remark: Checking the radius of convergence with the root test,

$$R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{1 \big/ |1-2i|^{n+1}}} = \lim_{n\to\infty} |1-2i|^{\frac{n+1}{n}} = |1-2i| = \sqrt{5}$$

EX 6: Expand $f(z) = \dfrac{3-i}{1-i+z}$ in a Taylor series with center $z_0 = 4-2i$.

Sol:

EX 7: Find the Taylor series and its convergence region of the following function with center $z_0 = 1$.

$$f(z) = \frac{2z^2 + 9z + 5}{z^3 + z^2 - 8z - 12}$$

Sol:

Laurent Series

Motivation

If a function *f*(z) fails to be analytic at a point z_0, one cannot apply Taylor's theorem at that point. Laurent series generalize Taylor series to find a series representation for *f*(z) involving both positive and negative powers of $z - z_0$.

Example

The function $f(z) = \dfrac{\sin z}{z^4}$ is not analytic at the isolated singularity $z = 0$ and hence cannot be expanded in a Maclaurin series. However, sin z is an entire function, we know that its Maclaurin series,

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots,$$

converges for $|z| < \infty$. By dividing this power series by z^4 we obtain a series for f with negative and positive integer powers of z:

$$f(z) = \frac{\sin z}{z^4} = \overbrace{\frac{1}{z^3} - \frac{1}{3!\,z}}^{\text{principal part}} + \overbrace{\frac{z}{5!} - \frac{z^3}{7!} + \frac{z^5}{9!} - \cdots}^{\text{analytic part}}.$$

The analytic part of the series converges for $|z| < \infty$. The principal part is valid for $|z| > 0$. Thus $f(z)$ converges for all z except at $z = 0$; that is, the series representation is valid for $0 < |z| < \infty$.

Laurent's Theorem

Theorem. *Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation*

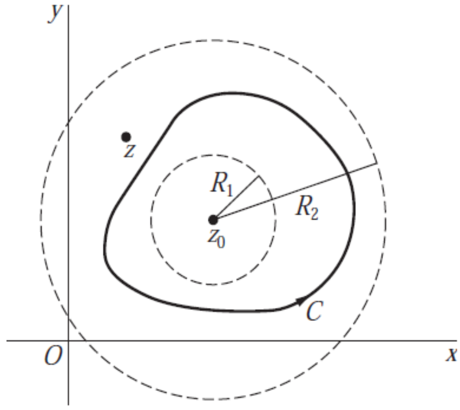
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad (R_1 < |z - z_0| < R_2),$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^{n+1}} \qquad (n = 0, 1, 2, \ldots)$$

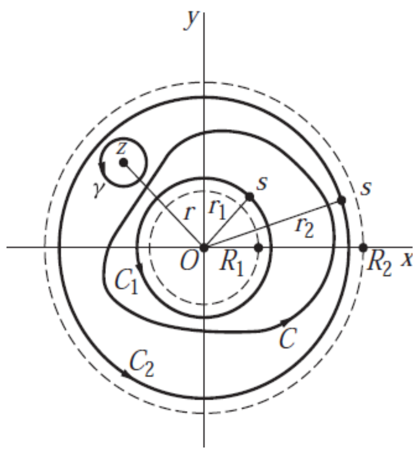
and

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^{-n+1}} \qquad (n = 1, 2, \ldots).$$



Proof of Laurent's Series

Proof:



Theoretical Method

Example: Find the power series of $f(z)=e^z/z^3$, $|z|>0$.

Sol:

Substitution Method

Although $f(z)=e^z/z^3$ is analytic for $|z|>0$, but is not analytic at $z=0$. Thus, the Maclaurin series does not exist from the theoretical method.

However, e^z for $|z|\geq 0$ has the Maclaurin series as

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$$

From uniqueness of power series, the representation of e^z/z^3 in power series can be obtained as

$$f(z)=\frac{1}{z^3}\cdot e^z=\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{2!z}+\frac{1}{3!}+\frac{z}{4!}+\frac{z^2}{5!}+\cdots$$

For all z such that $|z|>0$.

EX 1: Find the Laurent series of $z^2 e^{1/z}$ with center 0.

Sol:

EX: Find the Laurent series that represents the function

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$$

in the domain $0 < |z| < \infty$.

Ans. $1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}}$.

EX 2: Find the Laurent series of $1/(1-z)$ for (a) $|z|<1$ (b) $|z|>1$.

Sol:

EX 3: Find the Laurent series of $1/(z^3 - z^4)$.

Sol:

EX 4: Find the Laurent series of $f(z) = \dfrac{-2z+3}{z^2-3z+2}$.

Sol:

EX 5: Find the Laurent series of $f(z) = \dfrac{3}{2+z-z^2}$.

Sol:

EX 6: Find the Laurent series of $f(z)=\dfrac{z}{z^2-4z+3}$ in the region of $0<|z-1|<2$.

Sol:

EX 7: Find the Laurent series of $f(z)=\dfrac{z^2-2z+3}{z-2}$ in the region of $|z-1|>1$.

Sol: