Unit-4
Residue Integration

張大中
中央大學 通訊工程系
dcchang@ce.ncu.edu.tw

Zero of Order *n*

Definition

A number z_0 is zero of a function f if $f(z_0) = 0$. We say that an analytic function f has a zero of order n at $z = z_0$ if

$$\overbrace{f(z_0) = 0,\quad f'(z_0) = 0,\quad f''(z_0) = 0,\quad \ldots,\quad f^{(n-1)}(z_0) = 0}^{z_0 \text{ is a zero of } f \text{ and of its first } n-1 \text{ derivatives}},\quad \text{but}\quad f^{(n)}(z_0) \neq 0.$$

A zero of order n is also referred to as a zero of multiplicity n.

Theorem

A function f that is analytic in some disk $|z - z_0| < R$ has a zero of order n at $z = z_0$ if and only if f can be written

$$f(z) = (z - z_0)^n \phi(z),$$

where ϕ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$.

EX: Determine the order of zero at $z=0$.

$$f(z) = z\sin z^2$$

Sol:

Pole of Order *n*

Theorem

A function f analytic in a punctured disk $0 < |z - z_0| < R$ has a pole of order n at $z = z_0$ if and only if f can be written

$$f(z) = \frac{\phi(z)}{(z - z_0)^n},$$

where ϕ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$.

Corollary

If the functions g and h are analytic at $z = z_0$ and h has a zero of order n at $z = z_0$ and $g(z_0) \neq 0$, then the function $f(z) = g(z)/h(z)$ has a pole of order n at $z = z_0$.

EX 1: Locate the poles of $g(z)=\dfrac{1}{5z^4+26z^2+5}$ and specify their order.

Sol:

EX 2: Locate the poles of $g(z)=\dfrac{\pi\cot(\pi z)}{z^2}$ and specify their order.

Sol:

Residue

Definition

If $f(z)$ has a singularity at $z=z_0$ inside C but is otherwise analytic on C and inside C. Then $f(z)$ has a Laurent series

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n+\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\cdots$$

that converges for all points near $z=z_0$, in some domain $0<|z-z_0|<R$.

The coefficient b_1 is called the **residue** of $f(z)$ at $z=z_0$. Recall that

$$b_n=\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}dz,\quad n=1,2,3,\cdots$$

we have

$$\begin{aligned} b_1 &= \frac{1}{2\pi i}\oint_C f(z)dz \\ &= \operatorname*{Res}_{z=z_0} f(z) \end{aligned}$$

Note: also notation as $b_1=Res[f(z),z_0]$.

EX 1: Integrate $f(z) = z^{-4} \sin z$ counterclockwise around the unit circle.
Sol:

EX 2: Integrate $f(z) = \dfrac{1}{z^3 - z^4}$ clockwise around the circle $C: |z| = \dfrac{1}{2}$.
Sol:

EX 3: Integrate $f(z) = ze^{3/z}$ counterclockwise around the circle $C: |z| = 4$.
Sol:

Residue at a Simple Pole

Theorem

If f has a simple pole at $z = z_0$, then

$$\operatorname{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Proof:

Residue at a Pole of Order n

Theorem

If f has a pole of order n at $z = z_0$, then

$$\operatorname{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z).$$

Proof:

EX 1: Compute the residues at the singularities of

$$f(z) = \frac{1}{(z-1)^2(z-3)}$$

Sol:

EX 2: Compute the residues at the singularities of

$$f(z) = \frac{\cos z}{z^2(z-\pi)^3}.$$

Sol:

Residue at a Simple Pole

Theorem

Suppose a function $f(z)$ can be written as a quotient $f(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are analytic at $z = z_0$. If $p(z_0)\neq 0$ and if the function $q(z)$ has a simple zero at z_0 , then $f(z)$ has a simple pole at $z = z_0$ and

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Proof:

Example: $f(z) = \dfrac{9z+i}{z^3+z}$. Find $\operatorname*{Res}_{z=i} f(z)$.

Ans: $-5i$.

EX 1: Compute the residue at each singularity of $f(z) = \cot z$.

Sol:

EX 2: : Compute the residue at each singularity of $f(z) = \dfrac{1}{z^4+1}$.

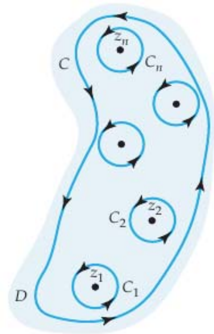
Sol:

Cauchy's Residue Theorem

Theorem

Let D be a simply connected domain and C a simple closed contour lying entirely within D. If a function f is analytic on and within C, except at a finite number of isolated singular points $z_1, z_2, \ldots, z_n$ within C, then

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k).$$



Proof:

EX 1: Evaluate $\oint_C \frac{2z+6}{z^2+4}\,dz$, where the contour C is the circle $|z-i| = 2$.

Sol:

EX 2: Evaluate $\oint_C \frac{e^z}{z^4+5z^3}\,dz$, where the contour C is the circle $|z| = 2$.

Sol:

EX 3: Evaluate $\oint_C \tan z \, dz$, where the contour C is the circle $|z| = 2$.

Sol:

EX 4: Evaluate $\oint_C \frac{\tan z}{z^2 - 1} dz$ in the counterclockwise sense where $C : |z| = \frac{3}{2}$.

Sol:

EX 5: Evaluate $\oint_C \frac{e^z - 1}{z^5} dz$ in the counterclockwise sense where C is the unit circle.

Sol:

Trigonometric Integration

Consider the following integrals

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta$$

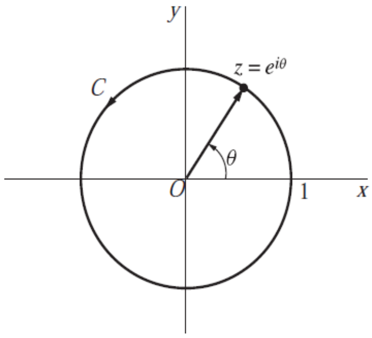
The basic idea here is to convert the real trigonometric integral into a complex integral, where the contour C is the unit circle |z| = 1 centered at the origin.

Let $z = e^{i\theta} \qquad (0 \le \theta \le 2\pi)$

$$\frac{dz}{d\theta} = ie^{i\theta} = iz \qquad d\theta = \frac{dz}{iz}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}$$



We have that

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta = \int_c F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) \frac{dz}{iz}$$

EX 1: Evaluate $\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta}$.

Sol:

Exercise: Evaluate $\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$.

Ans: $\frac{2\pi}{3}$.

EX 2: Evaluate $\int_0^{2\pi} \frac{d\theta}{1+3\cos^2\theta}$.

Sol:

Improper Integral

Def: Improper integral $f(x)$ over [0, ∞) is defined by

$$\int_0^\infty f(x)\,dx = \lim_{R\to\infty}\int_0^R f(x)\,dx.$$

provided that the limit exists. Similarly,

$$\int_{-\infty}^0 f(x)\,dx = \lim_{R\to\infty}\int_{-R}^0 f(x)\,dx.$$

If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$

provided both integrals are convergent (limit exists).

Note: If $\int_{-\infty}^{\infty} f(x)$ converges,

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{R\to\infty}\int_{-R}^R f(x)\,dx.$$

However, the symmetric limit may exists even though the improper integral $\int_{-\infty}^{\infty} f(x)$ is divergent.

Ex:

$\int_{-\infty}^{\infty} x\,dx$ is divergent since $\lim_{R\to\infty}\int_0^R x\,dx = \lim_{R\to\infty}\frac{1}{2}R^2 = \infty$.

$$\lim_{R\to\infty}\int_{-R}^R x\,dx = \lim_{R\to\infty}\frac{1}{2}[R^2-(-R)^2] = 0.$$

Cauchy Principal Value

Let $f(x)$ be a continuous real-valued function for all x. The **Cauchy principal value** (P.V.) of the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined by

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx,$$

provided the limit exists.

Example: Find P.V. $\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$.

Sol:

$$\begin{aligned}\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^2+1}dx &= \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2+1}dx \\ &= \lim_{R\to\infty} [\text{Arctan}\, R - \text{Arctan}\,(-R)] \\ &= \frac{\pi}{2} - \frac{-\pi}{2} = \pi.\end{aligned}$$

Cauchy Principal Value of the Integral of Rational Functions

Theorem

Let $f(z) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials of degree m and n, respectively. If $Q(x) \neq 0$ for all real x and $n \geq m+2$, then

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx = 2\pi i \sum_{j=1}^{K} \operatorname*{Res}_{z=z_j} f(z)$$

where $z_1, z_2, \cdots, z_K$ are the poles of $f(z)$ that lie in the upper half-plane.

Proof·

EX 1: Evaluate $\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$.

Sol:

EX 2: Evaluate $\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3}$.

Sol:

Jordan's Lemma in Upper Half-Plane

Theorem

Suppose that P and Q are polynomials of degree m and n, respectively, where $n \geq m+1$. If C_R is the upper semicircle $z = Re^{i\theta}$ for $0 \leq \theta \leq \pi$, then for $\alpha > 0$,

$$\lim_{R\to\infty} \int_{C_R} e^{i\alpha z} \frac{P(z)}{Q(z)} dz = 0$$

Proof:

Jordan's Lemma in Lower Half-Plane

Theorem

Suppose that P and Q are polynomials of degree m and n, respectively, where $n \geq m+1$. If C_R^- is the **lower** semicircle $z = Re^{i\theta}$ for $-\pi \leq \theta \leq 0$, then for $\alpha > 0$,

$$\lim_{R\to\infty} \int_{C_R^-} e^{-i\alpha z} \frac{P(z)}{Q(z)} dz = 0$$

Proof:

Fourier Integrals

Corollary

Let P and Q are polynomials of degree m and n, respectively. If $Q(x) \neq 0$ for all real x and $n \geq m+1$, then

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{i\alpha x} dx = 2\pi i \sum_{j=1}^{K} \operatorname*{Res}_{z=z_j} \left[\frac{P(z)}{Q(z)} e^{i\alpha z} \right]$$

That is,

$$\text{P.V.} \left[\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(\alpha x) dx + i \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(\alpha x) dx \right]$$
$$= 2\pi i \left(\text{Re}\left\{ \sum_{j=1}^{K} \operatorname*{Res}_{z=z_j} \left[\frac{P(z)}{Q(z)} e^{i\alpha z} \right] \right\} + i \, \text{Im}\left\{ \sum_{j=1}^{K} \operatorname*{Res}_{z=z_j} \left[\frac{P(z)}{Q(z)} e^{i\alpha z} \right] \right\} \right)$$

We have

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(\alpha x) dx = -2\pi \sum_{j=1}^{K} \text{Im}\left\{ \operatorname*{Res}_{z=z_j} \left[\frac{P(z)}{Q(z)} e^{i\alpha z} \right] \right\}$$

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(\alpha x) dx = 2\pi \sum_{j=1}^{K} \text{Re}\left\{ \operatorname*{Res}_{z=z_j} \left[\frac{P(z)}{Q(z)} e^{i\alpha z} \right] \right\}$$

where $\alpha > 0$ and $z_1, z_2, \cdots, z_K$ are the poles of $P(z)/Q(z)$ that lie in the upper half-plane.

EX 1: Evaluate $\text{P.V.}\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+1}dx$.

Sol:

EX 4: Evaluate $\text{P.V.}\int_{-\infty}^{\infty}\frac{\cos x}{x+i}dx$.

Sol:

(Method #1)

(Method #2)

Note, in this example,

$$\text{p.v.}\int_{-\infty}^{\infty}\frac{\cos x}{x+i}\,dx \neq \text{Re p.v.}\int_{-\infty}^{\infty}\frac{e^{ix}}{x+i}\,dx\ .$$

Improper Integrals

Definition

Suppose $t_1, t_2, \cdots, t_L$ are discontinuous points on the x-axis for $f(x)$, then

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)dx = \lim_{\substack{R\to\infty \\ \epsilon\to 0}} \sum_{j=1}^{L+1} \int_{t_{j-1}+\epsilon}^{t_j-\epsilon} f(x)dx$$

where $t_0 = -R$ and $t_{L+1} = R$.

Example: Evaluate $\text{P.V.}\int_1^4 \frac{dx}{x-2}$

Sol:

$$\int_1^{2-r}\frac{dx}{x-2}+\int_{2+r}^{4}\frac{dx}{x-2} = \text{Log}\,|x-2|\Big|_{x=1}^{x=2-r} + \text{Log}\,|x-2|\Big|_{x=2+r}^{x=4}$$

$$= \text{Log}\,r - \text{Log}\,1 + \text{Log}\,2 - \text{Log}\,r$$

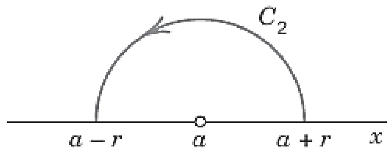
$$= \text{Log}\,2.$$

Integral of Indented Contour

Theorem

If $f(z)$ has a simple pole at $z=a$ on the real axis, then

$$\lim_{r\to 0}\int_{C_2} f(z)dz = \pi i \operatorname*{Res}_{z=a} f(z)$$



Proof:

Integral of Indented Contour of Rational Functions

Theorem

Let $f(z)=\dfrac{P(z)}{Q(z)}$, where P and Q are polynomials of degree m and n, respectively, and $n \geq m+2$.

If $Q(x)\neq 0$ and has simple zeros at the points $t_1, t_2, \cdots, t_L$ on the x-axis, then

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}dx = 2\pi i\sum_{j=1}^{K}\operatorname*{Res}_{z=z_j} f(z) + \pi i \sum_{j=1}^{L}\operatorname*{Res}_{z=t_j} f(z)$$

where $z_1, z_2, \cdots, z_K$ are the poles of $f(z)$ that lie in the upper half-plane.

Proof:

EX 1: Evaluate $\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)}$.

Sol:

Integral of Indented Contour of Rational Functions

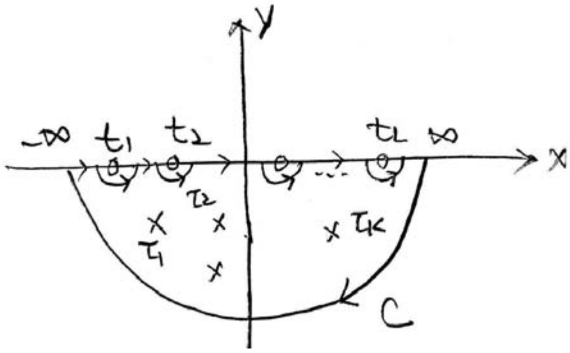
Corollary

Let $f(z)=\frac{P(z)}{Q(z)}$, where P and Q are polynomials of degree m and n, respectively, and $n \geq m+2$.

If $Q(x) \neq 0$ and has simple zeros at the points $t_1, t_2, \cdots, t_L$ on the x-axis, then

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = -2\pi i \sum_{j=1}^{K} \operatorname*{Res}_{z=\tau_j} f(z) - \pi i \sum_{j=1}^{L} \operatorname*{Res}_{z=t_j} f(z)$$

where $\tau_1, \tau_2, \cdots, \tau_K$ are the poles of $f(z)$ that lie in the lower half-plane.



Fourier Integral of Indented Contour of Rational Functions

Corollary

Let $f(z)=\dfrac{P(z)}{Q(z)}$, where P and Q are polynomials of degree m and n, $n \geq m+1$, respectively.

Let $Q(x) \neq 0$ and have simple zeros at the points $t_1, t_2, \cdots, t_L$ on the x-axis. If α is a positive real number, then

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{i\alpha x} dx = 2\pi i \sum_{j=1}^{K} \operatorname*{Res}_{z=z_j} f(z) e^{i\alpha z} + \pi i \sum_{j=1}^{L} \operatorname*{Res}_{z=t_j} f(z) e^{i\alpha z}$$

That is,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(\alpha x) dx = -2\pi \sum_{j=1}^{K} \text{Im}\left[\operatorname*{Res}_{z=z_j} f(z) e^{i\alpha z}\right] - \pi \sum_{j=1}^{L} \text{Im}\left[\operatorname*{Res}_{z=t_j} f(z) e^{i\alpha z}\right]$$

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(\alpha x) dx = 2\pi \sum_{j=1}^{K} \text{Re}\left[\operatorname*{Res}_{z=z_j} f(z) e^{i\alpha z}\right] + \pi \sum_{j=1}^{L} \text{Re}\left[\operatorname*{Res}_{z=t_j} f(z) e^{i\alpha z}\right]$$

where $z_1, z_2, \cdots, z_K$ are the poles of $f(z)$ that lie in the upper half-plane.

EX 1: Evaluate $\text{P.V.} \int_{-\infty}^{\infty} \frac{x e^{i2x}}{x^2-1} dx$.

Sol:

EX 2: Evaluate $\text{P.V.}\int_{-\infty}^{\infty}\frac{\sin x}{x(x^2-2x+2)}dx$.

Sol:

Integration Along a Branch Cut

Motivation: Since the integration involving z^α is a multiple-valued function, we can force z^α to be single valued for $z = re^{i\theta}$ by restricting θ to some interval of length 2π. We use the branch of the logarithm $\log_0$ as

$$z^\alpha = e^{\alpha \ln z} = e^{\alpha(\ln r + i\theta)}$$

where $z \neq 0$ and $0 < \theta \leq 2\pi$ is a branch of z^α.

Theorem

Let $f(z) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials of degree m and n, respectively, and $n \geq m+2$. If $Q(x) \neq 0$ for $x>0$ and $Q(x)$ has a zero of order at most 1 at the origin, and $0 < \alpha < 1$, then

$$\text{P.V.}\int_0^\infty \frac{x^\alpha P(x)}{Q(x)}dx = \frac{2\pi i}{1-e^{i2\alpha\pi}}\sum_{j=1}^{K}\operatorname*{Res}_{z=z_j}\left(z^\alpha f(z)\right)$$

where $z_1, z_2, \cdots, z_K$ are the nonzero poles of $f(z)$.

Proof:

EX 1: Evaluate $\text{P.V.} \int_0^\infty \frac{x^{-\alpha}}{1+x} dx,\ 0<\alpha<1$.

Sol:

EX 3: Evaluate $\text{P.V.} \int_0^\infty \frac{x^{-\alpha}}{x-4} dx,\ 0<\alpha<1$.

Sol:

EX 5: Evaluate $\text{P.V.} \int_0^\infty \frac{x^\alpha}{(x^2+1)^2}dx = \frac{(1-\alpha)\pi}{4\cos\left(\frac{\alpha\pi}{2}\right)}, \ -1<\alpha<3$

Sol:

Argument Principle

Theorem

Let C be a simple closed contour lying entirely within a domain D. Suppose f is analytic in D except at a finite number of poles inside C, and that $f(z) \neq 0$ on C. Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = Z_f - P_f,$$

where Z_f is the number of zeros of f that lie inside C and P_f is the number of poles of f that lie inside C.

Proof:

EX 1: Evaluate $\oint_C f'(z)/f(z)dz$ where $C:|z|=4$ is positively oriented.

$$f(z)=\frac{(z-8)^2 z^3}{(z-5)^4(z+2)^2(z-1)^5}$$

Sol:

EX: Evaluate $\oint_C f'(z)/f(z)dz$ where $C:|z|=\frac{3}{2}$ is positively oriented.

$$f(z)=\frac{(z-3iz-2)^2}{z(z^2-2z+2)^5}$$

Ans: $-18\pi i$

Laplace Transform

Definition

Let $f(t)$ be a real function and s be a complex variable. The Laplace transform of $f(t)$ is defined as

$$F(s)=\int_0^\infty e^{-st}f(t)dt$$

and is denoted as $\mathscr{L}\{f(t)\}$. The corresponding inverse pair is $f(t)=\mathscr{L}^{-1}\{F(s)\}$.

Example

The Laplace transform of $f(t)=1,\ t\geq 0$ is

$$\mathscr{L}\{1\}=\int_0^\infty e^{-st}(1)\,dt=\lim_{b\to\infty}\int_0^b e^{-st}\,dt$$
$$=\lim_{b\to\infty}\frac{-e^{-st}}{s}\bigg|_0^b=\lim_{b\to\infty}\frac{1-e^{-sb}}{s}. \qquad (5)$$

If s is a complex variable, $s=x+iy$, then recall

$$e^{-sb}=e^{-bx}(\cos by+i\sin by). \qquad (6)$$

From (6) we see in (5) that $e^{-sb}\to 0$ as $b\to\infty$ if $x>0$. In other words, (5) gives $\mathscr{L}\{1\}=\frac{1}{s}$, provided $\mathrm{Re}(s)>0$.

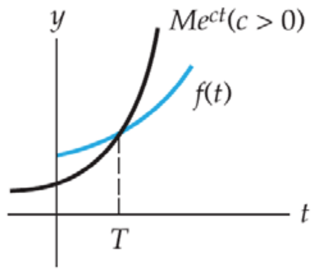
Exponential Order c

Definition

A function f is said to be **exponential order c** if there exist constants $c>0$, $M>0$, and $T>0$ so that $|f(t)| < Me^{ct}$, for $t>T$.

Remark 1: $e^{-ct}|f(t)|$ is bounded; that is, $e^{-ct}|f(t)| < M$ for $t>T$.

Remark 2: The condition $|f(t)| < Me^{ct}$ for $t>T$ states that the graph of f on the interval (T,∞) does not grow faster than the graph of the exponential function Me^{ct}.



Sufficient Conditions for Existence of Laplace Transform

Theorem

Suppose f is piecewise continuous on $[0, \infty)$ and of exponential order c for $t > T$. Then $\mathscr{L}\{f(t)\}$ exists for $\text{Re}(s) > c$.

Proof:

$$\mathscr{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt = I_1 + I_2.$$

The integral I_1 exists since it can be written as a sum of integrals over intervals on which $e^{-st}f(t)$ is continuous.

To prove the existence of I_2, we let s be a complex variable $s = x + iy$.

$|e^{-st}| = |e^{-xt}(\cos yt - i\sin yt)| = e^{-xt}$ and $|f(t)| \le Me^{ct}$, $t > T$,

$$|I_2| \le \int_T^\infty |e^{st} f(t)|\,dt \le M\int_T^\infty e^{-xt}e^{ct}dt$$

$$= M\int_T^\infty e^{-(x-c)t}dt = -M\frac{e^{-(x-c)t}}{x-c}\Bigg|_T^\infty = M\frac{e^{-(x-c)T}}{x-c}$$

for $x = \text{Re}(s) > c$.

Since $\int_T^\infty Me^{-(x-c)t}dt$ converges, this implies that I_2 exists for $\text{Re}(s) > c$.

Table of Laplace Transform

(i) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ [Re(s) > Re(a)]

(ii) $\mathcal{L}\{1\} = \mathcal{L}\{e^{0t}\} = \frac{1}{s}$ [Re(s) > 0]

(iii) $\mathcal{L}\{\cos \omega t\} = \text{Re}\,\mathcal{L}\{e^{i\omega t}\} = \frac{s}{s^2+\omega^2}$ [ω real, Re(s) > 0]

(iv) $\mathcal{L}\{\sin \omega t\} = \text{Im}\,\mathcal{L}\{e^{i\omega t}\} = \frac{\omega}{s^2+\omega^2}$ [ω real, Re(s) > 0]

(v) $\mathcal{L}\{\cosh \omega t\} = \mathcal{L}\{\cos i\omega t\} = \frac{s}{s^2-\omega^2}$ [ω real, Re(s) > $|\omega|$]

(vi) $\mathcal{L}\{\sinh \omega t\} = \mathcal{L}\{-i \sin i\omega t\} = \frac{\omega}{s^2-\omega^2}$ [ω real, Re(s) > $|\omega|$]

(vii) $\mathcal{L}\{e^{-\lambda t}\cos \omega t\} = \text{Re}\,\mathcal{L}\{e^{(-\lambda+i\omega)t}\} = \frac{s+\lambda}{(s+\lambda)^2+\omega^2}$
[ω, λ real, Re(s) > $-\lambda$]

(viii) $\mathcal{L}\{e^{-\lambda t}\sin \omega t\} = \text{Im}\,\mathcal{L}\{e^{(-\lambda+i\omega)t}\} = \frac{\omega}{(s+\lambda)^2+\omega^2}$
[ω, λ, real, Re(s) > $-\lambda$]

(ix) $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$ [Re(s) > Re(a)]

(x) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ [Re(s) > 0]

(xi) $\mathcal{L}\{t \cos \omega t\} = \text{Re}\,\mathcal{L}\{te^{i\omega t}\} = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$ [ω real, Re(s) > 0]

(xii) $\mathcal{L}\{t \sin \omega t\} = \text{Im}\,\mathcal{L}\{te^{i\omega t}\} = \frac{2s\omega}{(s^2+\omega^2)^2}$ [ω real, Re(s) > 0]

(xiii) $\mathcal{L}\{F(t)e^{-at}\}(s) = \mathcal{L}\{F\}(s+a)$

(xiv) $\mathcal{L}\{aF(t)+bH(t)\} = a\mathcal{L}\{F(t)\} + b\mathcal{L}\{H(t)\}$

Proof of Laplace Transform Pairs:

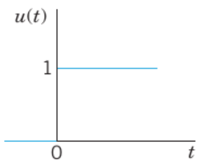
$$\mathcal{L}\{F(t)e^{-at}\}(s) = \int_0^\infty F(t)e^{-at}e^{-st}\,dt = \int_0^\infty F(t)e^{-(s+a)t}\,dt = \mathcal{L}\{F\}(s+a).$$

Laplace Transform of Time-Shift Functions

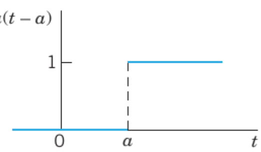
Definition

The **unit step function** or **Heaviside function** $u(t-a)$ is 0 for $t < a$, has a jump of size 1 at $t = a$ (where we can leave it undefined), and is 1 for $t > a$, in a formula:

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \quad (a \geqq 0).$$



Unit step function $u(t)$



Unit step function $u(t-a)$

$$\mathscr{L}\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)\,dt = \int_a^\infty e^{-st}\cdot 1\,dt = -\frac{e^{-st}}{s}\bigg|_{t=a}^{\infty} = \frac{e^{-as}}{s} \quad (s > 0)$$

Laplace Transform of Time-Shift Functions

If $\mathscr{L}\{f(t)\} = F(s)$, then $\mathscr{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$.

Proof:

Laplace Transform of the Derivative and Integral

- Laplace Transform of the Derivative:

By looking at the transform of the derivative $F'(t)$,

$$\mathcal{L}\left\{F'\right\}(s) = \int_0^\infty e^{-st} F'(t)\, dt$$

$$= -\int_0^\infty (-s)e^{-st} F(t)\, dt + e^{-st} F(t)\Big|_0^\infty$$

assuming that $e^{-st}F(t) \to 0$ as $t \to \infty$,

$$\mathcal{L}\left\{F'\right\}(s) = s\mathcal{L}\{F\}(s) - F(0).$$

Iterating this equation results in

$$\mathcal{L}\left\{F''\right\}(s) = s\mathcal{L}\left\{F'\right\}(s) - F'(0)$$
$$= s^2\mathcal{L}\{F\}(s) - sF(0) - F'(0),$$

and, in general,

$$\mathcal{L}\left\{F^{(k)}\right\}(s) = s^k\mathcal{L}\{F\}(s) - s^{k-1}F(0) - s^{k-2}F'(0) - \cdots - F^{(k-1)}(0).$$

- Laplace Transform of Integral:

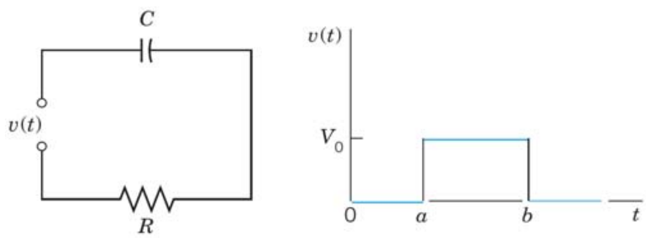
$$\mathscr{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}F(s)$$

Proof:

$$g(t) = \int_0^t f(\tau)\, d\tau\,,\ g'(t) = f(t),\ g(0) = 0$$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{g'(t)\} = s\mathscr{L}\{g(t)\} - g(0) = s\mathscr{L}\{g(t)\}.$$

Ex 1: Find the current $i(t)$ in the RC-circuit if a single rectangular wave with voltage V_0 is applied.



Solution:

EX 2: Find the function $f(t)$ that satisfies

$$\frac{d^2 f(t)}{dt^2} + 2\frac{df(t)}{dt} + f(t) = \sin t$$

for $t \geq 0$ and which at $t = 0$ has the properties $f(0) = 1,\ f'(0) = 0$.

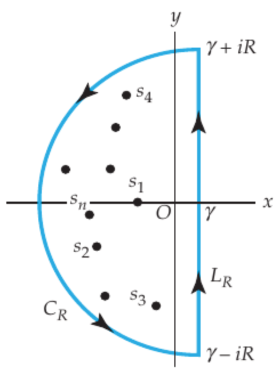
Sol:

Inverse Laplace Transform

Theorem (Mellin's Inverse Formula)

If f and f' are piecewise continuous on $[0, \infty)$ and f is of exponential order c for $t \geq 0$, and $F(s)$ is a Laplace transform, then the **inverse Laplace transform** $\mathscr{L}^{-1}\{F(s)\}$ is

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\gamma-iR}^{\gamma+iR} e^{st}F(s)\,ds,$$



where $\gamma > c$. Suppose $F(s)$ has a finite number of poles $s_1, s_2, \ldots, s_n$ to the left of the vertical line $\text{Re}(s) = \gamma$ and $sF(s)$ is bounded as $R \to \infty$, then

$$\mathscr{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \text{Res}\left(e^{st}F(s), s_k\right).$$

Remark:

The fact that $F(s)$ has singularities $s_1, s_2, \ldots, s_n$ to the left of the line $x = \gamma$ makes it possible for us to evaluate $\mathscr{L}^{-1}\{F(s)\}$ by using an appropriate closed contour encircling the singularities.

Proof:

EX 1: Evaluate $\mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\}$, $\mathrm{Re}(s) > 0$.

Sol:

Note: $\mathscr{L}\{t^n\} = n!/s^{n+1}$

EX 2: Evaluate $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{(s-1)(s-3)}\right\}$, $\text{Re}(s) > 3$.

Sol:

Note:

$$f(t) = \begin{cases} -\frac{1}{2}e^{t-2} + \frac{1}{2}e^{3(t-2)}, & t > 2 \\ 0, & t < 2. \end{cases}$$

$$= -\tfrac{1}{2}e^{t-2}\mathcal{U}(t-2) + \tfrac{1}{2}e^{3(t-2)}\mathcal{U}(t-2).$$

unit step function

$$\mathcal{U}(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$$

EX 3: Find the piecewise smooth function with Laplace transform $1/(s^4 - 1)$.

Sol:

Definition of Fourier Transform and Inverse Fourier Transform

Definition

Let $f(t)$ be a real function defined on the interval $(-\infty, \infty)$ and ω is a real variable. The **Fourier Transform** of $f(t)$ is defined as

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt$$

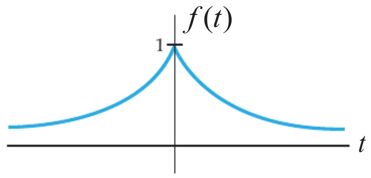
and the **inverse Fourier Transform** is

$$f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t} d\omega$$

Fourier Transform

Example: Find the Fourier transform of $f(t) = e^{-|t|}$.

Sol:



Fourier Transform of the Derivative

Theorem

Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$. Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then

$$\mathcal{F}\{f'(x)\} = iw\mathcal{F}\{f(x)\}.$$

Proof:

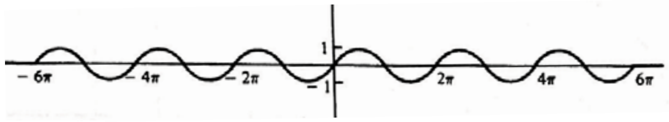
Inverse Fourier Transform

Ex1: Find the inverse Fourier transform of $F(\omega) = \dfrac{1}{\pi(1+\omega^2)}$.

Sol:

Ex2: Find the Fourier transform of the function and confirm the inversion formula.

$$F(t)=\begin{cases}\sin t, & |t|\le 6\pi,\\ 0, & \text{otherwise}\end{cases}$$



Sol:

Ex3: Find a function that satisfies the differential equation

$$\frac{d^2 f(t)}{dt^2}+2\frac{df(t)}{dt}-3f(t)=\begin{cases}1, & |t|<1\\ 0, & \text{otherwise}\end{cases}$$

Sol:

Numerical Validation for **Ex3**: