A New Variable Tap-Length and Step-Size FxLMS Algorithm

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***Abstract*—The filtered-X least mean-square (FxLMS) algorithm is widely used for active noise control (ANC). A long tap-length is usually required for some FxLMS applications, and consequentially the convergence rate becomes very slow. In this letter, a new variable tap-length and step-size FxLMS algorithm is proposed, especially suited to a long tap-length filter. Taking into account the lowpass filter effect in the secondary path of ANC, the new algorithm is developed for the control filter with an unsymmetric and two-sided exponential decay envelop over its impulse response. Simulation results show that the new algorithm provides faster convergence and cancellation performance compared to previously proposed variable-step-size FxLMS algorithms.**

***Index Terms*—Active noise control, FxLMS, secondary path, two-sided exponential decay envelop.**

I. Introduction

FILTERED-X least mean-square (FxLMS) is the most popular algorithm in the application of active noise cancellation (ANC) in terms of good noise reduction performance and low implementation cost [1]. Owing to the existence of a secondary path between the reference microphone and the error microphone in an ANC system, the FxLMS algorithm is employed to compensate for the secondary path effects in order to cancel the primary noise [2], [3]. Although the FxLMS algorithm used for ANC has some variants such as the lattice ANC, frequency-domain ANC, delayless subband ANC, etc. [1], the transversal filter structure is relatively simple from the aspect of implementation.

In conventional FxLMS algorithms, the performance and convergence properties have been well studied for the use of a fixed step size. Taking into consideration some practical circumstances, the required tap length of the control filter is usually so long that a fixed-step-size FxLMS algorithm may lead to slow convergence or large excess mean-square error. Some modified LMS algorithms such as the normalized LMS and correlation LMS are possible candidates that can improve the convergence rate [1]. In addition to attaining fast convergence, the variable-step-size FxLMS algorithm [4], [5] recently drew a lot of attention because it can cope with both stationary and nonstationary environments.

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In practical applications, the tap length of the primary plant is unknown, and using excess tap length for the control filter will lead to an increase of misadjustment in the LMS algorithm. In this letter, we propose a new variable tap-length and step-size FxLMS algorithm to improve the convergence rate and performance. In the literature, there are some existing variable-tap-length LMS algorithms [6]–[8] that consider a constant exponential decay envelop for the impulse response of the plant in a system identification model. For ANC applications, the secondary path usually includes a lowpass filter, resulting in a two-sided decay envelop over the impulse response. Moreover, the maximum output of the plant is not necessarily at the middle of its impulse response. Hence, we develop the new variable-tap-length FxLMS algorithm under the assumption of the control filter having an unsymmetric and two-sided exponential decay envelop over its impulse response, and also develop a new variable-step-size scheme ensuring global convergence. Numerical results will show that the new FxLMS algorithm has fast convergence and better steady-state mean-square deviation (MSD) performance compared to conventional FxLMS algorithms.

II. A Variable Tap-Length and Step-Size FxLMS Algorithm

A. System Model

The block diagram of a typical FxLMS algorithm is depicted in Fig. 1, in which $P(z)$ is an unknown plant and $W(z)$ is an adaptive filter used to compensate the secondary path $S(z)$ in order to cancel the disturbance. The background noise $v(n)$ is usually uncorrelated with $x(n)$ and adds to the cancellation error signal $e(n)$. The objective of $W(z)$ is to minimize $e(n)$. Denote the output of $P(z)$ as $d(n)$ and the impulse response of $S(z)$ as $s(n)$. Consider that K is a sufficient tap length for $W(z)$ such that the coefficient vector of $W(z)$ at time index n is $\mathbf{w}(n) = [w_0(n)\ w_1(n)\ \ldots\ w_{K-1}(n)]^T$ and the input noise vector $\mathbf{x}(n) = [x(n)\ x(n-1)\ \ldots\ x(n-K+1)]^T$. The cancellation error signal can be expressed as

$$e(n) = d(n) - [\mathbf{w}^T(n)\mathbf{x}(n)] * s(n) + v(n), \qquad (1)$$

where $*$ denotes linear convolution. The adaptive filter $\mathbf{w}(n)$ is then updated in the negative gradient direction with step size μ as $\mathbf{w}(n+1) = \mathbf{w}(n) - (\mu/2)\nabla\widehat{\xi}(n)$, where $\widehat{\xi}(n) = e^2(n)$ is an instantaneous estimate of the MSE gradient at time n. Accordingly, $\nabla\widehat{\xi}(n) = -2\mathbf{x}'(n)e(n)$, where $\mathbf{x}'(n) = s(n) * \mathbf{x}(n) = [x'(n)\, x'(n-1)\, \ldots\, x'(n-K+1)]^T$, then

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \mu\mathbf{x}'(n)e(n). \qquad (2)$$

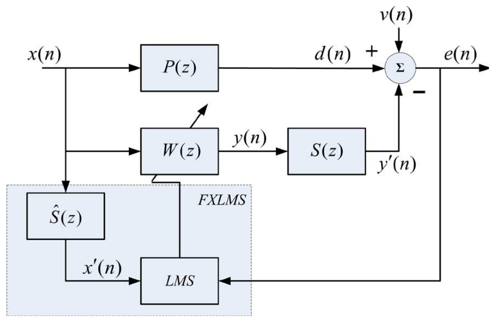


Fig. 1. Block diagram of the FxLMS algorithm.

From (2), $S(z)$ is included in the update equation of the adaptive filter coefficients, and is conventionally called the filtered-X LMS or FxLMS algorithm. In general, $S(z)$ is usually unknown, and therefore, $x'(n) = \widehat{s}(n) * x(n)$, where $\widehat{s}(n)$ is an on-line estimated impulse response of $S(z)$ [2]. Here, we simply treat $\widehat{S}(z) = S(z)$ in the following work.

B. Proposed Algorithm

The decay envelops of the impulse responses of $P(z)$ and $S(z)$ are possibly presented on both sides of the maximum output response. For example, $S(z)$ usually contains a lowpass filter and $P(z)$ may have an unsymmetric decay envelop to account for plant propagation delays. From (1), the z-transform of the cancellation error signal is

$$E(z) = [P(z) - W(z)S(z)]X(z) + V(z). \tag{3}$$

Ignoring $V(z)$, a simple insight into (3) is that the cancellation error approaches zero, i.e., $E(z) = 0$, after the adaptive filter converges. Hence, we can see that the optimal $W(z)$ is to realize

$$W^o(z) = \frac{P(z)}{S(z)}. \tag{4}$$

Here, we generally consider that the impulse response of $W(z)$ has an unsymmetric decay envelop, where the left tap length of the maximum response is M while the right tap length is N, with the optimal coefficients written as $\mathbf{w}_K^o = [w_{-M}^o \cdots w_{-1}^o \; w_0^o \; w_1^o \cdots w_{N-1}^o]^T$, where w_0^o denotes the maximum response. For simplicity, the following exponential function is used to model the envelop of the impulse response coefficients of $\mathbf{w}_K^o$:

$$w_k^o = \begin{cases} e^{k\tau_1} r_w(k), & k = -M, -M+1, \ldots, -1, \\ e^{-k\tau_2} r_w(k), & k = 0, 1, \ldots, N-1, \end{cases} \tag{5}$$

where the decay factors τ_1 and τ_2 are positive constants, and $r_w(k)$ is a zero-mean i.i.d. Gaussian random sequence with variance $\sigma_{r_w}^2$. Neglecting those coefficients of the exponentially decayed magnitude less than 1/10, we can choose, at least, $M = \lfloor 2.3/\tau_1 \rfloor$ and $N = \lfloor 2.3/\tau_2 \rfloor$. Actually, using somewhat larger M and N in the algorithm does not significantly affect the performance.

The proposed FxLMS algorithm adaptively adjusts its tap length and step size as time progresses. Denote by $L(n)$, $R(n)$, and $\mu(n)$ the left-hand-side tap length, right-hand-side tap length, and step size at time n, respectively. We let $K(n) = L(n) + R(n)$, where $K(n) \le K$. Using the notation $\mathbf{w}_{L(n),R(n)}(n)$ and $\mathbf{x}'_{L(n),R(n)}(n)$ with subscripts $L(n)$ and $R(n)$ to represent the $K(n)$-tap filter vector and input vector, respectively, where $\mathbf{w}_{L(n),R(n)}(n) = [w_{-L(n)}(n) \; \cdots \; w_{-1}(n) \; w_0(n) \; w_1(n) \; \cdots \; w_{R(n)-1}(n)]^T$ and $\mathbf{x}'_{L(n),R(n)}(n) = [x'(n + L(n)) \cdots x'(n+1) \, x'(n) \, x'(n-1) \cdots x'(n - R(n) + 1)]^T$, we can rewrite (2) as

$$\mathbf{w}_{L(n+1),R(n+1)}(n+1) = \begin{bmatrix} \mathbf{0}_{L(n+1)-L(n)} \\ \mathbf{w}_{L(n),R(n)}(n) \\ \mathbf{0}_{R(n+1)-R(n)} \end{bmatrix} + \mu(n+1)e(n) \times \mathbf{x}'_{L(n+1),R(n+1)}(n). \tag{6}$$

where $\mathbf{0}_j$ denotes the $j \times 1$ zero vector.

To express the vector $\mathbf{w}_K(n)$ of $W(z)$ at time n by the modeled part $\mathbf{w}_{L(n),R(n)}(n)$, we write $\mathbf{w}_K(n) = [\mathbf{0}_{M-L(n)}^T \; \mathbf{w}_{L(n),R(n)}^T(n) \; \mathbf{0}_{N-R(n)}^T]^T$. Now, split $\mathbf{w}_K^o$ into three parts as $\mathbf{w}_K^o = [\mathbf{w}_{M-L(n)}^{oT} \; \mathbf{w}_{L(n),R(n)}^{oT} \; \mathbf{w}_{N-R(n)}^{oT}]^T$, so the total coefficient error is

$$\mathbf{g}_K(n) = \mathbf{w}_K(n) - \mathbf{w}_K^o. \tag{7}$$

From (4), we can express the output of $P(z)$ as

$$d(n) = [\mathbf{w}_K^{oT} \mathbf{x}_K(n)] * s(n). \tag{8}$$

Substituting (8) into (1) and using (7), the cancellation error signal becomes

$$\begin{aligned} e(n) &= [\mathbf{w}_K^{oT} \mathbf{x}_K(n) - \mathbf{w}_{L(n),R(n)}^T(n) \mathbf{x}_{L(n),R(n)}(n)] \\ &\quad * s(n) + v(n) \\ &= -\,[\mathbf{g}_K^T(n)\mathbf{x}_K(n)] * s(n) + v(n) \\ &= -\,\mathbf{g}_K^T(n)\mathbf{x}'_K(n) + v(n). \end{aligned} \tag{9}$$

Substituting (9) for $e(n)$ in (6) and subtracting $\mathbf{w}_K^o$ on both sides of (6), we obtain

$$\mathbf{g}_K(n+1) = \mathbf{A}(n)\mathbf{g}_K(n) + \mu(n+1)v(n) \begin{bmatrix} \mathbf{0}_{M-L(n+1)} \\ \mathbf{x}'_{L(n+1),R(n+1)}(n) \\ \mathbf{0}_{N-R(n+1)} \end{bmatrix}, \tag{10}$$

where

$$\mathbf{A}(n) = \mathbf{I}_K - \mu(n+1) \begin{bmatrix} \mathbf{0}_{M-L(n+1)} \\ \mathbf{x}'_{L(n+1),R(n+1)}(n) \\ \mathbf{0}_{N-R(n+1)} \end{bmatrix} \mathbf{x}'^T_K(n), \tag{11}$$

and $\mathbf{I}_K$ is the $K \times K$ identity matrix.

To develop the recursive algorithm for $L(n+1)$, $R(n+1)$, and $\mu(n+1)$, the MSD of $\mathbf{g}_K(n)$ can be explored by expressing

$$\Lambda(n) \equiv E[\|\mathbf{g}_K(n)\|_2^2], \tag{12}$$

where $\|.\|_2$ denotes ℓ_2 norm and $E[\cdot]$ represents taking expectation. Assume that $x(n)$ and $v(n)$ are i.i.d. Gaussian sequences with variances σ_x^2 and σ_v^2, respectively. According to a similar assumption and analysis in [6], we have

$$\Lambda(n+1) = \eta\Lambda(n) + (\beta - \eta)\Gamma(n+1) + \gamma, \quad (13)$$

where

$$\Gamma(n+1) = E\left[\left\|\mathbf{w}^o_{M-L(n+1)}\right\|_2^2\right] + E\left[\left\|\mathbf{w}^o_{N-R(n+1)}\right\|_2^2\right], \quad (14)$$

$$\eta = 1 - 2\mu(n+1)\sigma_{x'}^2 + [K(n+1)+2]\mu^2(n+1)\sigma_{x'}^4, \quad (15)$$

$$\beta = 1 + K(n+1)\mu^2(n+1)\sigma_{x'}^4, \quad (16)$$

$$\gamma = K(n+1)\mu^2(n+1)\sigma_{x'}^2\sigma_v^2, \quad (17)$$

and using (5) for $\mathbf{w}^o_K$, we have

$$E[\|\mathbf{w}^o_{M-L(n+1)}\|_2^2] = \frac{1-e^{2[M-L(n+1)]\tau_1}}{1-e^{2M\tau_1}}E[\|\mathbf{w}^o_M\|_2^2], \quad (18)$$

$$E[\|\mathbf{w}^o_{N-R(n+1)}\|_2^2] = \frac{e^{-2R(n+1)\tau_2} - e^{-2N\tau_2}}{1-e^{-2N\tau_2}}E[\|\mathbf{w}^o_N\|_2^2], \quad (19)$$

where

$$E[\|\mathbf{w}^o_M\|_2^2] = \frac{e^{-2M\tau_1}\left(1-e^{2M\tau_1}\right)}{1-e^{2\tau_1}}\sigma_{r_w}^2, \quad (20)$$

$$E[\|\mathbf{w}^o_N\|_2^2] = \frac{1-e^{-2N\tau_2}}{1-e^{-2\tau_2}}\sigma_{r_w}^2. \quad (21)$$

The optimal tap length can be found by minimizing the MSD with respect to $L(n+1)$ and $R(n+1)$. Taking the partial derivatives of $\Lambda(n+1)$ with respect to $L(n+1)$ and $R(n+1)$ and setting to zero, we obtain, after some mathematical manipulation,

$$L(n+1) = -\frac{1}{2\tau_1}\ln\frac{\mu(n+1)\sigma_e^2(1-e^{2\tau_1})}{-4\tau_1\sigma_{r_w}^2[1-\mu(n+1)\sigma_{x'}^2]}, \quad (22)$$

$$R(n+1) = -\frac{1}{2\tau_2}\ln\frac{\mu(n+1)\sigma_e^2(1-e^{-2\tau_2})}{4\tau_2\sigma_{r_w}^2[1-\mu(n+1)\sigma_{x'}^2]}, \quad (23)$$

where $\sigma_e^2 = \sigma_{x'}^2\Lambda(n) + \sigma_v^2$, obtained from (9).

The optimal $\mu(n+1)$ can be found in a similar manner. However, $\mu(n+1)$ becomes related to $L(n+1)$ and $R(n+1)$, so it is difficult to get a closed-form solution for the joint equations. Making the quasi-static assumptions $L(n) \approx L(n+1)$ and $R(n) \approx R(n+1)$, a suboptimal solution can be efficiently found by replacing $L(n+1)$ and $R(n+1)$ by $L(n)$ and $R(n)$, respectively, to calculate $\mu(n+1)$. Finally, we have

$$\mu(n+1) = \frac{1 - \frac{\Gamma(n)}{\Lambda(n)}}{[K(n)+2]\sigma_{x'}^2 + \frac{[K(n)\sigma_v^2 - 2\sigma_{x'}^2\Gamma(n)]}{\Lambda(n)}}. \quad (24)$$

From (24), we can observe that when $v(n)$ is ignored and the adaptive filter approaches the perfect tap length, $\sigma_v^2 = \Gamma(n) = 0$, and thus $\mu(n+1)$ approaches $1/(K+2)\sigma_{x'}^2$.

C. Global Convergence

Returning to (13), if it can be proved that the second-order derivatives of $\Lambda(n+1)$ with respect to $L(n+1)$, $R(n+1)$, and $\mu(n+1)$ are positive, the MSD is a convex function of $L(n+1)$, $R(n+1)$, and $\mu(n+1)$, and therefore, the recursions will find the minimum MSD. Taking the second-order derivatives of (13) with respect to $L(n+1)$, $R(n+1)$, and $\mu(n+1)$, respectively, we have

$$\frac{\partial^2\Lambda(n+1)}{\partial L^2(n+1)} = \frac{-8\mu(n+1)\tau_1^2\sigma_{x'}^2\sigma_{r_w}^2[1-\mu(n+1)\sigma_{x'}^2]}{e^{2L(n+1)\tau_1}(1-e^{2\tau_1})}, \quad (25)$$

$$\frac{\partial^2\Lambda(n+1)}{\partial R^2(n+1)} = \frac{8\mu(n+1)\tau_2^2\sigma_{x'}^2\sigma_{r_w}^2[1-\mu(n+1)\sigma_{x'}^2]}{e^{2R(n+1)\tau_2}(1-e^{-2\tau_2})}, \quad (26)$$

$$\frac{\partial^2\Lambda(n+1)}{\partial \mu^2(n+1)} = 2K(n+1)\sigma_{x'}^2\sigma_e^2(n) + 4\sigma_{x'}^4[\Lambda(n) - \Gamma(n)]. \quad (27)$$

In (25) and (26), $1-\mu(n+1)\sigma_{x'}^2 > 0$ because $\mu(n+1)$ is usually smaller than 10^{-3} with a normalized power of $x'(n)$. From (7) and (12), we have that $\Lambda(n) - \Gamma(n) = E[\|\mathbf{g}_{K(n)}(n)\|_2^2]$. Therefore, in (27), $\Lambda(n) - \Gamma(n) \geq 0$. From the above results, (25)–(27) are all positive such that the MSD is a convex function of $L(n+1)$, $R(n+1)$, and $\mu(n+1)$. That is, we have proved that the new FxLMS algorithm can converge to the minimum MSD.

III. Simulation Results

Two numerical experiments are employed to compare the performance of different FxLMS algorithms. For the first experiment, $W^o(z)$ is known and generated from a zero-mean white Gaussian process $r_w(k)$ with $\sigma_{r_w}^2 = 0.01$, $\tau_1 = 0.01$, and $\tau_2 = 0.005$. The tap length is $K = 1024$ which is divided into $M = 124$ and $N = 900$. The characteristic of $S(z)$ is similar to [4] with 65 taps. For simplicity, we let $\widehat{S}(z)$ equal $S(z)$ here and generate $P(z) = W^o(z)S(z)$. Once the estimate of $W^o(z)$ is obtained as $\widehat{W^o}(z)$, the MSD is evaluated through a 100-run Monte Carlo simulation by calculating $\|\hat{\mathbf{w}}^o(n) - \mathbf{w}^o_K(n)\|_2^2$. The reference noise and the background noise are zero-mean i.i.d. and uncorrelated Gaussian processes with variances $\sigma_x^2 = 1$ and $\sigma_v^2 = 0.01$, respectively.

In Fig. 2, the MSD performance is shown in decibels (dB). Since the tap length of $W^o(z)$ is 1024, we choose the same tap length for conventional FxLMS algorithms, where the largest step size, according to the description below (24), is set as $\mu_{\max} = 1/(1024+2) \times \sigma_{x'}^2$ and the smallest step size $\mu_{\min} = 0.2\mu_{\max}$, which in 50000 iterations provides steady-state performance close to that of the proposed algorithm. In addition to the normalized FxLMS algorithm [1], a variable step-size FxLMS algorithm, similar to [4], is simulated using

$$\mu(n) = \rho(n)\mu_{\max} + [1-\rho(n)]\mu_{\min}, \quad (28)$$

where $\rho(n)$ is a weighting factor, $0 \leq \rho(n) \leq 1$, and is calculated by $\rho(n) = P_e(n) - P_{e,\min}/P_{e,\max} - P_{e,\min}$ with $P_{e,\max}$ and $P_{e,\min}$ representing the maximum and minimum average powers

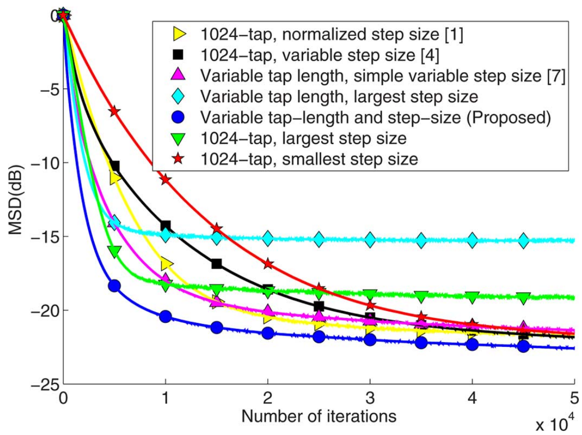
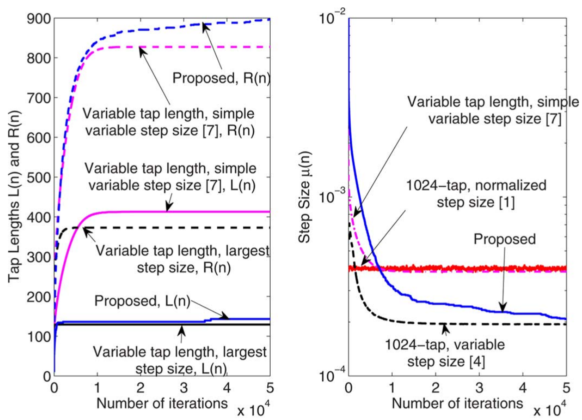


Fig. 2. Comparison of MSD for different FxLMS algorithms.



(a) (b)

Fig. 3. Convergence comparison of tap length and step size for proposed algorithm and other FxLMS algorithms. (a) Tap lengths $L(n)$ and $R(n)$. (b) Step size $\mu(n)$.

of $e(n)$, respectively, and $P_e(n) = (1/T)\sum_{i=n-T+1}^{n} e^2(n)$, where T is an averaging constant with $T = 200$ here. $P_{e,\max}$ is estimated using the average of the first 100 iterations of $P_e(n)$ multiplied by 1.3, while $P_{e,\min}$ using the average of the last 100 iterations of $P_e(n)$ multiplied by 0.7. The proposed algorithm with (24) achieves excellent convergence speed and MSD performance. However, if we consider the simple variable step size used in [7] for the new variable-tap-length method with $\mu(n) = 0.5/[L(n) + R(n) + 65]\sigma_{x'}^2$, the convergence rate becomes worse since the step size is still lacking flexibility compared to the proposed algorithm. When the step size $\mu_{\max}$ is used with the new variable-tap-length method, the MSD performance significantly degrades in spite of a fast convergence rate. Fig. 3(a) and (b) show the convergence curves of tap lengths and step sizes with the proposed and other FxLMS algorithms, respectively. To prevent from overestimating the tap lengths, the proposed algorithm terminates the growth of tap lengths when the ℓ_2 norms of the new 20 taps are less than 10^{-2} for $L(n)$ and 10^{-4} for $R(n)$, providing that the total tap length approaches 1024 and the step size remains a larger value for faster convergence compared to the variable step size method [4].

For the second experiment, assume that $P(z)$ and $S(z)$ are known with characteristics similar to the data included in [1] with tap lengths 145 and 51, respectively. The decay factors used in this case are $\tau_1 = 0.09$ and $\tau_2 = 0.04$ such that $M = 25$ and $N = 90$ are set in our algorithm. Since the tap lengths are shorter

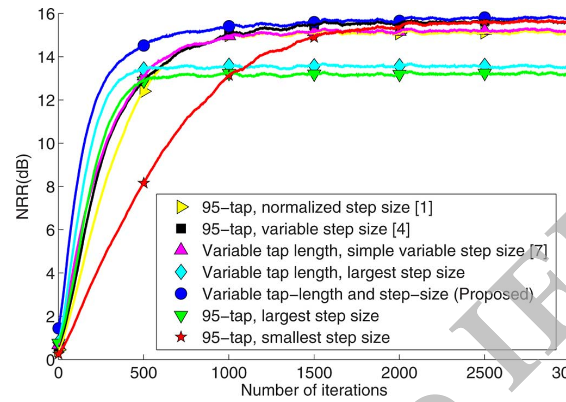


Fig. 4. Comparison of NRR for different FxLMS algorithms.

than those used in the first experiment, the ℓ_2 norms of only the last 10 new taps are calculated to terminate the tap growth. The noise variances are the same as those used in the first experiment. Here, the noise reduction performance is evaluated instead of the MSD. We define the noise reduction ratio NRR (dB) as

$$\mathrm{NRR(dB)} = 10\log\left(\frac{E[d^2(n)]}{E[e^2(n)]}\right), \quad (29)$$

where for simplicity, $E[\cdot]$ can be implemented by ensemble average. Fig. 4 compares the NRR for different algorithms. The proposed algorithm also has superior NRR performance because of its fast convergence and minimum MSD properties.

IV. Conclusions

Normalized and variable step-size LMS algorithms are conventionally tailored for FxLMS. Since the plant and secondary path models are usually unknown, the parameter setup of FxLMS is almost heuristic, so that it is not easy to find a good tradeoff between convergence rate and performance. In contrast, the proposed algorithm shows a promising result with self-adjusting tap lengths, especially suited to a long-tap-length design. Under the assumption of a two-sided exponential decay envelop, the new algorithm even results in minimum MSD.

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